Arbitrage Pricing Theory and Muti-Factor Model

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1 Overview

In this note, we introduce the arbitrage pricing theory (APT) in Ross (1976). Where Ross suggests a muti-factor model only depends on the no arbitrage assumption. As comparisons, CAPM is derived from diversification and modern portfolio theory, and Lucas (1978) uses equilibrium concept to derive stochastic discount factor. However, we will also see that, CAPM is a special case in APT model.

2 Arbitrage Pricing Theory

- Assume a linear factor model for the return on a set of assets:

$$R_i = \mu_i + \beta_{i,1} f_1 + \beta_{i,2} f_2 + ... + \epsilon_i, i = 1, 2, ..., n. \tag{1}$$

  Where $E[f_k] = 0$, $E[\epsilon_i] = 0$, and $E[\epsilon_i, \epsilon_j] = 0, \forall i \neq j$. R_i is the asset return, μ_i is the expected asset return, β_i is the factor loading or exposure, and f is the risk factor.

- For simplicity, we now first discuss one factor model,

$$R_i = \mu_i + \beta_i f + \epsilon_i, i = 1, 2, ..., n. \tag{2}$$

  It can also be rewritten in matrix,

$$\vec{R} = \vec{\mu} + \vec{\beta} f + \vec{\epsilon}. \tag{3}$$

- Now we will construct an arbitrage portfolio which has zero investment, assume the face value of each asset today is 1 unit, we should have

$$\vec{\omega}' \vec{1} = 0. \tag{4}$$

- Then the return of this portfolio is

$$R_A = \vec{\omega}'\vec{\mu} + \vec{\omega}'\vec{\beta}f + \vec{\omega}'\vec{\epsilon}, \tag{5}$$

if this arbitrage portfolio is well diversified and $E[\epsilon_i, \epsilon_j] = 0$, $\vec{\omega}'\vec{\epsilon}$ will be close to 0.

- Besides, an arbitrage portfolio should have zero exposure to factor,

$$\vec{\omega}'\vec{\beta} = 0 \tag{6}$$

- Equation (4) and (6) implies that the choices set of $\vec{\omega}'$ is orthogonal to the panel spanned by $\vec{1}$ and $\vec{\beta}$.

- In summary, this arbitrage portfolio has zero investment, and has zero risk exposure. So that it must have zero return,

$$R_A = \vec{\omega}'\vec{\mu} = 0. \tag{7}$$

Such that $\vec{\mu}$ is in the same panel spanned by $\vec{1}$ and $\vec{\beta}$:

$$\vec{\mu} = \gamma_1\vec{1} + \gamma_2\vec{\beta}. \tag{8}$$

- If we choose two special assets: the risk-free asset and market portfolio.

- Assume the factor f is market risk factor $R_M - \mu_M$, then we know the exposure β_M for market portfolio is 1.[1]

$$R_f = \gamma_1 \tag{9}$$

$$R_M = \gamma_1 + \gamma_2 \tag{10}$$

$$\Rightarrow \gamma_1 = R_f, \gamma_2 = \mu_M - R_f \tag{11}$$

- In other words, any asset's expected return is:

$$\mu_i = R_f + \beta_i(\mu_M - R_f). \tag{12}$$

- This is exactly CAPM said. That is why we said CAPM is a special case in APT if people believe the demeaned market return $R_M - \mu_M$ is a risk factor. The factor model will become

$$\mu_i = R_f + \beta_{i,M}(\mu_M - R_f) \tag{13}$$

[1] It is trivial that $R_M = \mu_M + 1 \times [R_M - \mu_M]$.

3 Discussion

- APT model is a statistical model, it has no restriction on the factor once it satisfies $E[f_k] = E[\epsilon_i] = E[\epsilon_i, \epsilon_j] = 0$.
- We can expand it to multi-factor model, if we believe there is also a risk factor related with Bitcoin defined as $f_2 = R_{BTC} - \mu_{BTC}$. Then the multi-factor model will be

$$\mu_i = R_f + \beta_{i,M}(\mu_M - R_f) + \beta_{i,BTC}(\mu_{BTC} - R_f) \tag{14}$$

- Or generally,

$$\mu_i = R_f + \beta_{i,1}\lambda_1 + \beta_{i,2}\lambda_2 + ... \tag{15}$$

where βs are still factor exposure, and λ is risk premium of factor.
- We can also easily find that, ICAPM is also under this framework,

$$E(R) - R_f = \beta_m(\mu_m - R_f) + \sum_{i=1}^{n} \beta_i^h(\mu_i^h - R_f). \tag{16}$$

4 Question

- A natural question is that whether FF3 (1993) is also under this framework?
- We already know that FF3 can be equivalent to SDF approach in my previous note[2] if we believe SDF is linear to SMB.
- In the same note, we also prove that APT is equivalent to SDF approach if we believe SDF is linear to risk premium (e.g. $\mu_M - R_f$).[3]
- Unfortunately, I don not find a way to unite them together. For instance, if we use similar logic in section 3 to set $f_3 = SMB - \mu_{SMB}$, then we will get

$$\mu_i = R_f + \beta_{i,M}(\mu_M - R_f) + \beta_{i,SMB}(\mu_{SMB} - R_f) \tag{17}$$

, which is incosistent with FF3 which only uses SMB in the second term.

[2] `https://github.com/ronming1303/Ruming-Liu-PDF-Document/blob/main/Factor_Model_and_Linear_Stochastic_Discount_Factor.pdf`

[3] In this case, the factor loading in SDF is $\beta_{SDF} = Cov(R_i - R_f, R_M - R_f)Var(R_M - R_f)^{-1} = Cov(R_i, R_M - \mu_M)Var(R_M - \mu_M)^{-1}$, which is also the $\beta_{i,SMB}$ in equation (13). SDF approach further says $\mu_i - R_f = \beta_{SDF}(\mu_M - R_f)$, which is equivalent to equation (13).

- What is wrong here? My answer is that the SMB is an arbitrage portfolio with 0 initial investment, its return in equation (1) will be infinite. It will be inappropriate to derive a similar result anymore. Another explanation for this question `https://quant.stackexchange.com/questions/58200/fama-french-vs-arbitrage-pricing-theory-of-ross`.

5 Reference

- Cochrane, J.H., 2009. Asset pricing: Revised edition. Princeton university press.
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