Arrow-Debreu(1954) Market

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1 Overview

This note is to summarize the core idea of Arrow-Debreu security and market which is introduced by Arrow, and Debreu (1954) in "Existence of an Equilibrium for a Competitive Economy".

2 Model Setup

- There are K individuals in the market.
- Individual k has initial endowment $e_{k,0}$ at time $t = 0$.
- At time $t = 1$, there are Ω different states in this world. For each state $\omega \in [1, 2, 3, ..., \Omega]$, individual k will have different endowment denoted as $e^i_{k,1}$.
- The initial endowment vector without financial market for individual k is,

$$\begin{bmatrix} e_{k,0} \\ e^1_{k,1} \\ e^2_{k,1} \\ \dots \\ e^\Omega_{k,1} \end{bmatrix} := \begin{bmatrix} e_{k,0} \\ \vec{e}_{k,1} \end{bmatrix} \tag{1}$$

- Individual k can arrange her consumption vector c_k as

$$c_k = \begin{bmatrix} c_{k,0} \\ c^1_{k,1} \\ c^2_{k,1} \\ \dots \\ c^\Omega_{k,1} \end{bmatrix} := \begin{bmatrix} c_{k,0} \\ \vec{c}_{k,1} \end{bmatrix} \tag{2}$$

- A typical Arrow-Debreu security on ω is a contingent claim at time $t = 1$, which pays 1 if ω happens, otherwise 0.

$$P_1^{\omega} = \begin{bmatrix} 0 \\ ... \\ 1 \\ ... \\ 0 \end{bmatrix} \tag{3}$$

- Assume the market is complete, each state can be covered by a corresponding A-D security, the total payoff matrix of the A-D market can be represented as a diagonal matrix

$$P_1 = \begin{bmatrix} 1 \ 0 \ ... \ 0 \\ 0 \ 1 \ ... \ 0 \\ ... \\ 0 \ 0 \ ... \ 1 \end{bmatrix}_{\Omega \times \Omega} \tag{4}$$

- We define the ω state price as the price of the A-D security which pays 1 when ω happens. The state prices of all A-D securities are vector:

$$\phi = \begin{bmatrix} \phi_1 \\ \phi_2 \\ ... \\ \phi_{\Omega} \end{bmatrix} \tag{5}$$

- The A-D market is complete. It is trivial that any future consumption $\vec{c}_{k,1}$ can be perfectly replicated by holding the portfolio $\vec{X}^{A-D}$ of A-D securities at time $t = 0$,

$$\vec{X}^{A-D} \cdot P_1 = \vec{c}_{k,1} \Rightarrow \vec{X}^{A-D} = \vec{c}_{k,1}. \tag{6}$$

With the cost at $t = 0$,

$$\phi^T \cdot \vec{c}_{k,1} \tag{7}$$

3 Solution

- What is the equilibrium under this market?

- For any initial endowment of individual k at time t,

$$\begin{bmatrix} e_{k,0} \\ \vec{e}_{k,1} \end{bmatrix} \tag{8}$$

- The value of her initial endowment can be calculated as

$$e_{k,0} + \phi^T \cdot \vec{e}_{k,1} \tag{9}$$

- She can rearrange her consumption to c_k by holding A-D portfolio, which costs

$$c_{k,0} + \phi^T \cdot \vec{c}_{k,1} \tag{10}$$

- The total costs should be less than total wealth:

$$\Rightarrow c_{k,0} + \phi^T \cdot \vec{c}_{k,1} <= e_{k,0} + \phi^T \cdot \vec{e}_{k,1}, \forall k \tag{11}$$

- The equilibrium can solved as a utility maximization problem:

$$\begin{aligned} &\max_{c_{k,0},\vec{c}_{k,1}} U_k(c_{k,0}, \vec{c}_{k,1}), \\ &s.t.\ c_{k,0} + \phi^T \cdot \vec{c}_{k,1} <= e_{k,0} + \phi^T \cdot \vec{e}_{k,1}, \end{aligned} \tag{12}$$

4 Discussion

Arrow-Debreu (1954) is one of the most important asset pricing papers, even though its concepts are straightforward. Using financial instruments (Arrow-Debreu securities) to re-arrange the payoff in different states (uncertainty).

It uses rigorous mathematical representation to describe the uncertainty, utility maximization, and equilibrium price in the financial market.

The limitation is that it only considers the game of single period, and with finite states.