Barra Pure Factor Portfolio

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February 7, 2025

1 Overview

This is a note of Barra China Equity Model (CNE5). We will introduce how the model puts forward a pure factor model from statistics.

2 Motivation of a 'Pure' Factor

- When we construct a portfolio exposed to some risk factor. Risk manager may asks two questions:
- Q1. How can we make sure that it is only exposed to this unique risk factor but not other risk factor?
- Q2. How can we make sure that the portfolio's risk exposure is 1?
- For the first question. If we can split other risk from portfolio, we will be very confident to say that our portfolio performance is compensated by the exposure of that risk factor.
- For the second question. If the portfolio's risk exposure is changing year by year, it is hard to compare the factor return of the same risk factor because the portfolio's exposure changes year by year.
- So we need to find a way to construct a portfolio that only has 1 unit exposure to a specific risk factor, but not other risk factors. Barra gives their solutions.

3 Barra's Pure Factor Model

- We summarize the CNE5[1] factor model which includes 1 country factor, P industry factors, and Q style factors. We denote $K = 1 + P + Q$.
- The original factor model for excess can be written as,

$$\boldsymbol{R}_t^e = \boldsymbol{\beta}_{t-1}\boldsymbol{\lambda}_t + \boldsymbol{u_t} \tag{1}$$

- The matrix $\boldsymbol{\beta}$ is the exposures of stocks to different factors. All stocks have 1 exposure to country factor. Specifically,

$$\boldsymbol{\beta}_{t-1} = \begin{bmatrix} 1 & \beta_{1,t-1}^{I_1} & ... & \beta_{1,t-1}^{I_P} & \beta_{1,t-1}^{S_1} & ... & \beta_{1,t-1}^{S_Q} \\ 1 & \beta_{2,t-1}^{I_1} & ... & \beta_{2,t-1}^{I_P} & \beta_{2,t-1}^{S_1} & ... & \beta_{2,t-1}^{S_Q} \\ ... & ... & ... & ... & ... & ... & ... \\ 1 & \beta_{N,t-1}^{I_1} & ... & \beta_{N,t-1}^{I_P} & \beta_{N,t-1}^{S_1} & ... & \beta_{N,t-1}^{S_Q} \end{bmatrix}_{[N\times K]} \tag{2}$$

- Our goal is to figure out the factor return $\boldsymbol{\lambda_t}$ given the observation of $\boldsymbol{R}_t^e$ and $\boldsymbol{\beta}_{t-1}$.
- The model assumes that each stock is assigned to only 1 industry factor β^I, to avoid the colinearity of country factor and industry factors. The model adds additional constraint to industry risk return:

$$S_{I_1}\lambda_{I_1,t} + S_{I_12}\lambda_{I_2,t} + ... + S_{I_P}\lambda_{I_P,t} = 0 \tag{3}$$

where S_{I_P} is the market capitalization weight of stocks in this industry.

- The constraint can be rewritten as matrix form:

$$\begin{bmatrix} \lambda_C \\ \lambda_{I_1} \\ ... \\ \lambda_{I_{P-1}} \\ \lambda_{I_P} \\ \lambda_{S_1} \\ ... \\ \lambda_{S_Q} \end{bmatrix}_{K\times 1} = \begin{bmatrix} 1 & 0 & 0 & ... & 0 & 0 & ... & 0 \\ 0 & 1 & 0 & ... & 0 & 0 & ... & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & ... & 1 & 0 & ... & 0 \\ 0 & \frac{-S_{I_1}}{S_{I_P}} & \frac{-S_{I_2}}{S_{I_P}} & ... & \frac{-S_{I_{P-1}}}{S_{I_P}} & 0 & ... & 0 \\ 0 & 0 & 0 & ... & 0 & 1 & ... & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & ... & 0 & 0 & ... & 1 \end{bmatrix}_{K\times K-1} \times \begin{bmatrix} \lambda_C \\ \lambda_{I_1} \\ ... \\ \lambda_{I_{P-1}} \\ \lambda_{S_1} \\ ... \\ \lambda_{S_Q} \end{bmatrix}_{K-1\times 1} \tag{4}$$

[1] https://www.msci.com/www/webcast/the-barra-china-equity-model/013761175

- For simplicity, we denote this constraint as:

$$\boldsymbol{\lambda} = \boldsymbol{C}\boldsymbol{\gamma} \tag{5}$$

- The model also normalized the exposure to style factors $\boldsymbol{\beta}^S$, in other words, $\beta_{t-1}^{S_q} \sim N(.), \forall q = 1, 2, ...Q$.

- For technique reasons (Check the reference), the model adds a weighted matrix for each stocks:

$$\boldsymbol{W} = \begin{bmatrix} \frac{\sqrt{S_1}}{\sum\sqrt{S_n}} & 0 & \cdots & 0 \\ 0 & \frac{\sqrt{S_2}}{\sum\sqrt{S_n}} & \cdots & 0 \\ \cdots & \cdots & \ddots & \cdots \\ 0 & 0 & \cdots & \frac{\sqrt{S_N}}{\sum\sqrt{S_n}} \end{bmatrix}_{N\times N} \tag{6}$$

- Combine regression equation (1), constraint equation (5), and weight matrix equation (6), we can get the weighted least square (WLS) estimation with constranit:

$$\hat{\boldsymbol{\lambda}} = \boldsymbol{C}(\boldsymbol{C}'\beta'\boldsymbol{W}\beta\boldsymbol{C})^{-1}\boldsymbol{C}'\beta'\boldsymbol{W}\boldsymbol{R}^e \tag{7}$$

- We denote:

$$\boldsymbol{\Omega} := \boldsymbol{C}(\boldsymbol{C}'\beta'\boldsymbol{W}\beta\boldsymbol{C})^{-1}\boldsymbol{C}'\beta'\boldsymbol{W} \tag{8}$$

- The economic meaning of $\boldsymbol{\Omega}$ is the weight be invested to individual stocks for replicating the corresponding risk factor.

- For instance, if we want to construct a portfolio with 1 exposure to I_1 factor, we just need to use $[\omega_{1,I_1}, \omega_{2,I_1}, ..., \omega_{N,I_1}]$ to construct the portfolio.

- But recall that equation (7) is the WLS estimator, which means the portfolio is very close to the theoretic portfolio which only has 1 exposure to I_1. But it may still have little exposures to other factors.

4 'Pure' Factor Portfolio

- In previous section, we derive the portfolio construction matrix in equation (8).

- We now investigate the risk exposures of each 'pure' factor portfolio.

- In other words, we investigate the characteristics of $\mathbf{\Omega}\boldsymbol{\beta}$, we can prove that

$$\mathbf{\Omega}\boldsymbol{\beta} = \begin{bmatrix} D_{(1+P)\times(1+P)} & 0_{(1+P)\times Q} \\ 0_{Q\times(1+P)} & I_{Q\times Q} \end{bmatrix} \tag{9}$$

- Exposures of 'pure' factor portfolio to risk factors:

- The 1st row of $\mathbf{\Omega}\boldsymbol{\beta}$ indicates:

  - Country factor portfolio is fully invested.

  - Country factor portfolio has 1 exposure to country factor and its construction weight is almost same to market portfolio.[2]

  - It has positive exposures to all industry factors.

  - It has 0 exposures to all style factors. $(0_{(1+P)\times Q})$

- The 2nd to $(1+P)$-th rows of $\mathbf{\Omega}\boldsymbol{\beta}$ indicates:

  - Industry factor portfolio is 0 invested.

  - Industry factor portfolio invests 100% to the stocks in this industry and short 100% to country factor portfolio. It has positive exposure (very closed to 1) to this industry factor, but negative (very closed to 0) to other industry factors.

  - It has 0 exposures to all style factors. $(0_{(1+P)\times Q})$

- The $(P+2)$-th to $(1+P+Q)$-th rows of $\mathbf{\Omega}\boldsymbol{\beta}$ indicates:

  - Style factor portfolio is 0 invested.

  - Style factor portfolio invests some stocks and short some stocks. It has 1 exposure to its style factor. But it has 0 exposure to country, industry, and other style factors. $(0_{Q\times(1+P)} \& I_{Q\times Q})$

5 Discussion

- The pure factor portfolio suggests a way to construct a 'clean' portfolio which is almost 'purely' exposed to specific risk factor.

- Ignoring the feasibility of using $\mathbf{\Omega}$ to construct portfolio.

- It still gives a way to track the performance of specific factor.

[2]Proof: `https://zhuanlan.zhihu.com/p/38280638`

- Besides, if we already know two returns can be decomposed to these factor returns, then known the pure factor return, we can calculate the correlation of those two returns indirectly.

6 Reference

- `https://zhuanlan.zhihu.com/p/38280638`
- `https://zhuanlan.zhihu.com/p/39922829`
- `https://zhuanlan.zhihu.com/p/653815618`
- Barra Risk Model Handbook (2007). MSCI.