Factor Model and Linear Stochastic Discount Factor

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March 13, 2025

1 Overview

In previous note, we have proved that CAPM, CCAPM, and Black Scholes Model are specific applications under some appropriate settings in SDF approach.

In this note, we will further prove a more general results, any linear factor model is actually a special case when SDF also has a linear structure, they are equivalent.[1]

2 Linear Stochastic Discount Factor

- In CAPM and CCAPM, the SDF is the marginal utility of tomorrow's consumption divided by marginal utility of today's assumption.
- But in real world, how to calculate this SDF depends on the utility function of all individuals. It may be very complicated with many non-linear terms.
- Why not assuming the SDF has linear structure? We assume the SDF has linear structure,

$$m = a + b'f. \tag{1}$$

- We can further demeaned the factors f

$$m = a + b'(f - E[f]), \tag{2}$$

and $E(m) = a$.

[1] Cochrane shows that an expected return - beta model is equivalent to a model for the discount factor that is a linear function of the factors in the beta model. We are following his proof.

- The connection is easiest to see in the special case that all the test assets are excess returns. Then $E(mR^e) = E(m(R - R_f)) = E(mR) - E(mR_f) = 0$. It does not identify the unique m, and we can normalize arbitrarily. It is convenient to normalize to $E(m) = 1$, such that

$$m = 1 + b'(f - E[f]) \tag{3}$$

$$E(mR^e) = 0 \tag{4}$$

- Trivially, we have

$$\begin{aligned} 0 &= E[mR^e] \\ &= E[m]E[R^e] + b'Cov(f, R^e) \\ &= E[R^e] + b'Cov(f, R^e) \end{aligned} \tag{5}$$

$$E[R^e] = -b'Cov[f, R^e] = -b'Var(f)Var(f)^{-1}Cov(f, R^e) \tag{6}$$

- Assume β are the multiple regression coefficients of excess return R^e on the factors

$$\beta = Var(f)^{-1}Cov(f, R^e) \tag{7}$$

- Equation (6) can be rewritten as,

$$E[R^e] = \beta'\lambda \tag{8}$$

  where $\lambda = -Var(f)b$

- "An expected return beta model is equivalent to a discount factor that is a linear function of the factors in the beta model. This is an important and central result. It gives the connection between the discount factor formulation and the expected return-beta factor model formulation common in empirical work."

- In general cases, the factor f can be anything, if we choose it as excess return (e.g. excess return of market portfolio in CAPM).

- Time f on both sides of (3) and take expectation, we have

$$E(mf) = E(f) + bVar(f) = 0 \tag{9}$$

- Because f is excess return, which also satisfies $E(mf) = 0$, such that

$$\lambda = -Var(f)b = E(f) \tag{10}$$

- In other words,
$$E(R^e) = \beta E(R_M^e), \tag{11}$$
where β is the exposure of systematic risk.

- Equation (11) is consistent with the CAPM formula. We have proved that linear factor models are equivalent to linear SDF.

- Furthermore, if we set $f = SMB/HML$ as the long short 0 investment portfolio in FF3, because it is zero investment, we still have $E(mf) = 0$, and similar result as equation (10), which leads the FF3 model,
$$E(R^e) = \beta E(R_M^e) + \beta_{SMB} E(SMB) + \beta_{HML} E(HML). \tag{12}$$

- Recall FF (1993), they run the regression and take expectation on both hand sides to get equation (12) by statistical method.

- Now we understand this from another angle: FF3 is equivalent with proxy SDF by linear regression with SMB and HML.

3 Reference

- Cochrane, J.H., 2009. Asset pricing: Revised edition. Princeton university press.

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