Stevens Institute of Technology

Game Theory Notes

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Contents

1 Strategic game and Nash equilibrium

1.1 Why we need to exclude the dominated strategies?

In a strategic game, every players are rational and smart, they know those strategies which will never been played will not be applied by any players. Those never been played strategies are defined as strictly dominated.

Remark

Definition 1. Pure strategy s_i is strictly dominated for player i if there exists $\sigma_i^{'} \in \Sigma_i$ s.t.

$$u_i(\sigma_i^{'}, s_{-i}) > u_i(s_i, s_{-i}), \forall s_{-i} \in S_{-i}. \tag{1}$$

In other words, playing s_i is not reasonable because for any other players' actions $s_{-i} \in S_{-i}$, there is always a better action $\sigma_i^{'}$ than s_i.

1.2 Nash equilibrium

Definition 2. A mixed-strategy profile σ^* is a Nash equilibrium if, for all players i,

$$u_i(\sigma_i^*, \sigma_{-i}^*) \geq u_i(s_i, \sigma_{-i}^*). \tag{2}$$

Note that a pure-strategy Nash equilibrium (NE) is a pure-strategy profile that satisfies the same conditions. We should also expect that if a player uses a mixed strategy in NE, he must be indifferent between all pure strategies to which he assigns positive probability. Otherwise, he should choose the dominant strategy among these pure strategies.

NE are "consistent" predictions of how the game will be played, in the sense that if all players predict that a particular NE will occur then no player has an incentive to play differently. If the NE is unique, then every players will reach to this equilibrium.

A NE is strict if each player has a unique best response to his rivals' strategies. That is, s^* is a strict equilibrium if and only if it is a NE and, for all i and all $s_i \neq s_i^*$,

$$u_i(s_i^*, s_{-i}^*) > u_i(s_i, s_{-i}^*).$$

By definition, a strict equilibrium is necessarily a pure-strategy equilibrium.

Lemma 1. When rounds of elimination of strictly dominated strategies yields a unique strategy profile $s^* = (s_1^*, ..., s_n^*)$, this strategy profile is the unique NE. This is because any strategy $s_i \neq s_i^*$ is strictly dominated by s_i^*. In particular,

$$u_i(s_i^*, s_{-i}^*) > u_i(s_i, s_{-i}^*) \forall i \in N.$$

1.3 Existence of Nash equilibrium

Theorem 1. Every finite strategic-form game has a mixed-strategy equilibrium.

Remark. *Kindly remark that a pure-strategy is a special mixed strategy, so pure-strategy NE is included in this assertion. But the absence of a pure-strategy equilibrium in some games should not be surprising, since pure-strategy equilibrium need not exist in finite games.*

Theorem 2. Consider a strategic-form game whose strategy spaces S_i are nonempty compact convex subsets of an Euclidean space. If the payoff functions u_i are continuous in s and quasi-concave in s_i, there exists a pure-strategy Nash equilibrium.

Remark. *Of course, NE can exist even when the conditions of the existence theorems are not satisfied, as these conditions are sufficient but not necessary.*

Theorem 3. Consider a strategic-form game whose strategy spaces S_i are nonempty compact subsets of a metric space. If the payoff functions u_i are continuous in s then there exists a NE in mixed strategies.

1.4 Iterated strict dominance

Definition 3. The process of iterated deletion of strictly dominated strategies proceeds as follows: Set $S_i^0 := S_i$ and $\Sigma_i^0 := \Sigma_i$. Now define S_i^n recursively by

$$S_i^n = \{s_i \in S_i^{n-1} | \text{there is no } \sigma_i \in \Sigma_i^{n-1} \; s.t. \; u_i(\sigma_i, s_{-i}) > u_i(s_i, s_{-i}), \forall s_{-i} \in S_{-i}\},$$

and define

$$\Sigma_i^n = \{\sigma_i \in \sigma_i | \sigma_i(s_i) > 0, \text{only if } s_i \in S_i^n\}.$$

Set

$$S_i^\infty := \cap_{n=0}^\infty S_i^n,$$

to represent the set of player i's pure strategies that survive iterated deletion of strictly dominated strategies. And set Σ_i^∞ to be all mixed strategies σ_i such that there is no $\sigma^{'i}$ with $u_i(\sigma_i^{'}, s_{-i}) > u_i(\sigma_i, s_{-i})$ for all $s_{-i} \in S_{-i}^\infty$. This is the set of player i's mixed strategies that survive iterated strict dominance.

Theorem 4. A game is solvable by iterated (strict) dominance if, for each player i, S_i^∞ is a singleton.

2 Extensive-Form games with incomplete information and equilibrium refinement

In the example of prisoners' dilemma, the players choose their actions simultaneously. But sometimes, games are played sequentially, game theory uses the concept of a game in extensive form to model such dynamic situations. Complete information means players can observe all the actions before their moves.

2.1 Information set

Definition 4. Every player i has a collection of information sets $h_i \in H_i$ that partition the nodes of the game at which player i moves with the following properties:

1. If h_i is a singleton that includes only one node then player i whoc moves at x knows that he is at x.

2. If $x \neq x'$ and if both $x \in h_i$ and $x' \in h_i$ then player i who moves at x does not know whether he is at x or x'.

3. If $x \neq x'$ and if both $x \in h_i$ and $x' \in h_i$ then $A_i(x') = A_i(x)$.

In figure 1, we use dotted line or box to represent the information set h_2, in this information set, the player 2 can't distinguish which node he is at. And he has the same strategy profiles at both possible nodes.

Definition 5. A game of complete information in which every information set is a singleton and there are no moves of Nature is called a game of **perfect information**. A game in which some information sets contain several nodes or in which there are moves of Nature is called a game of **imperfect information**.

We can easily see that strategic game is a game of imperfect information, it can be represented as a tree. Figure 1 is the tree to represent the battle of the sex game.

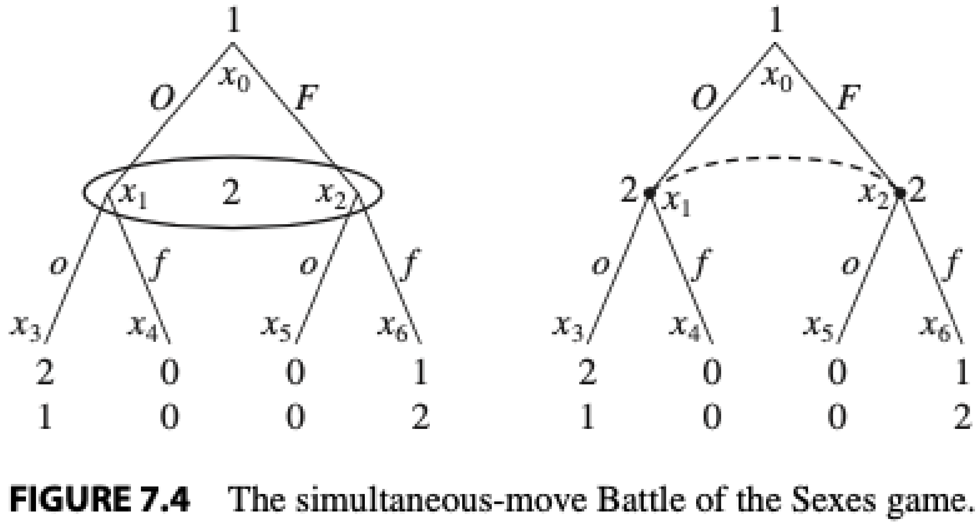

Figure 1: Battle of sex game in Tadlis Chapter 7.

2.2 Strategy profile in extensive-form games

In the strategic-form game, it is very easy to define a strategy profile for player, everyone only knows some common knowledge and everyone's information is symmetric. But in extensive-form game, players have asymmetric information. Besides common knowledge, they may decide their strategies based on what he just learnt, our strategy profile now should be defined on the information set.

Definition 6. A **pure strategy** for player i is a mapping $s_i : H_i \to A_i$ that assigns an strategy $s_i(h_i) \in A_i(h_i)$ for every information set $h_i \in H_i$. We denote by S_i the set of all pure-strategy mappings $s_i \in S_i$.

Definition 7. A **mixed strategy** for player i is a probability distribution over his pure strategies $s_i \in S_i$.

This definition of mixed strategy works fine in strategic-form game. But in extensive-form game, when players in different information sets, he may learn something from previous actions from other players. He wants to make mixed strategies based on information sets. To allow for strategies that let players randomize as the game unfolds we define a new concept as follows:

Definition 8. A **behavioral strategy** specifies for each information set $h_i \in H_i$ an independent probability distribution over $A_i(h_i)$ and is denoted by $\sigma_i : H_i \to \Sigma_i A_i(h_i)$, where $\sigma_i(a_i(h_i))$ is the probability that player i plays action $a_i(h_i) \in A_i(h_i)$ in information set h_i.

Remark. *Any extensive-form game can be transformed into a normal-form game by using the set of pure strategies of the extensive form as the set of pure strategies in the normal form, and the set of payoff functions is derived from how combinations of pure strategies result in the selection of terminal nodes. Furthermore every extensive-form game will have a unique normal form that represents it, which is not true for the reverse transformation.*

Clearly this exercise of transforming extensive-form games into the strategic form seems to miss the point of capturing the dynamic structure of the extensive-form game. Why then would we be interested in this exercise? The reason is that the concept of a Nash equilibrium is static in nature, in that the equilibrium posits that players take the strategies of others as given, and in turn they play a best response. Therefore the strategic-form representation of an extensive form will suffice to find all the NE of the game. This is particularly useful if the extensive form is a two-player game with a finite number of strategies for each player, because we can write its normal form as a matrix and solve it with the simple techniques developed earlier.

Definition 9. Let $\sigma^* = (\sigma_1^*, ...\sigma_n^*)$ be a NE profile of behavioral strategies in an extensive-form game. We say that an information set is **on the equilibrium path** if given σ^* it is reached with positive probability. We say that an information set if **off the equilibrium path** if given σ^* it is never reached.

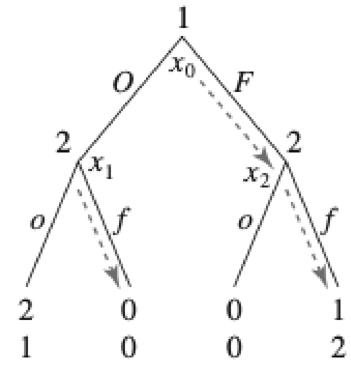

FIGURE 7.13 Equilibrium paths in the sequential-move Battle of the Sexes game.

Figure 2: Battle of sexes game in Tadlis Chapter 7.

In the game in figure 2, given the strategies profile $\{F, ff\}$, information sets x_0 and x_2 are on the equilibrium path, but x_1 is off the path.

2.3 The strategic-form representation of extensive-form games

An extensive-form game can also be represented by strategic-form matrix.

2.4 Sequential rationality and backward induction

Definition 10. Given strategies $\sigma_{-i} \in \Sigma_{-i}$ of i's opponents, we say that σ_i is sequential rational if and only if i is playing a best response to σ_{-i} in his information set, given his belief μ in that information set. In particular,

$$E[u_i|\mu, \sigma_i, \sigma_{-i}] \geq E[u_i|\mu, \hat{\sigma}_i, \sigma_{-i}]$$

In words, at each information set, the action taken by the player with move must be optimal given the player's belief at that information set and the other players' subsequent strategies.[1] Note that with complete information, players' information sets are singleton, the belief is unique to $\mathbb{P}(\text{at singleton}) = 1$.

Definition 11. Backward Induction: We can apply the logic of sequential rationality to any extensive game in the following way: Start at the "end" of the game tree, and work "back" up the tree by solving for sequential rationality at each node.

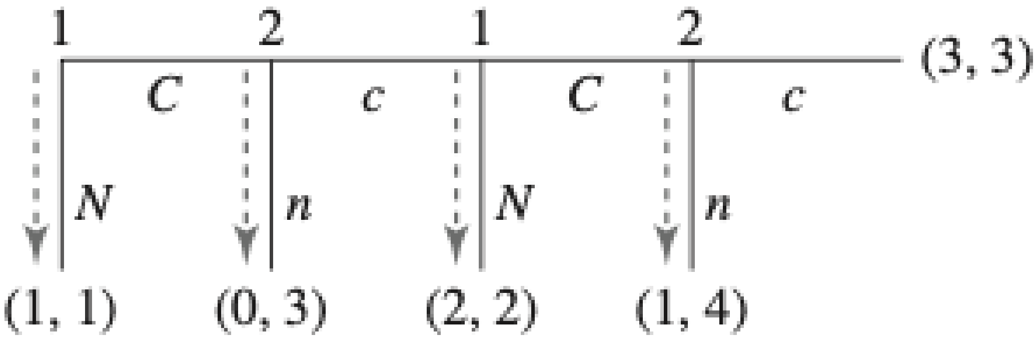

FIGURE 8.8 The Centipede Game.

Figure 3: Centiped Game in Tadlis Chapter 8.

In the centipede game in figure 3, we can apply backward induction to find the NE $\{NN, nn\}$, and the equilibrium payoff $(1, 1)$. Backward induction is a useful method to find a sequentially rational NE in finite games of perfect information. But things become a bit trickier when we try to expand our reach to suggest solutions for games of imperfect information, in which backward induction as previously defined encounters some problems.

In the game in figure 4, when player 2 is the start point of backward induction. If the information set is singleton, he can easily make his best choice. But player 2 is hard to make a sequential ra-

[1] This definition can be refered at here and here.

tional strategy if there is uncertainty at information set.[2] At that situation, we need to extend the concept of sequential rationality to games of imperfect information.

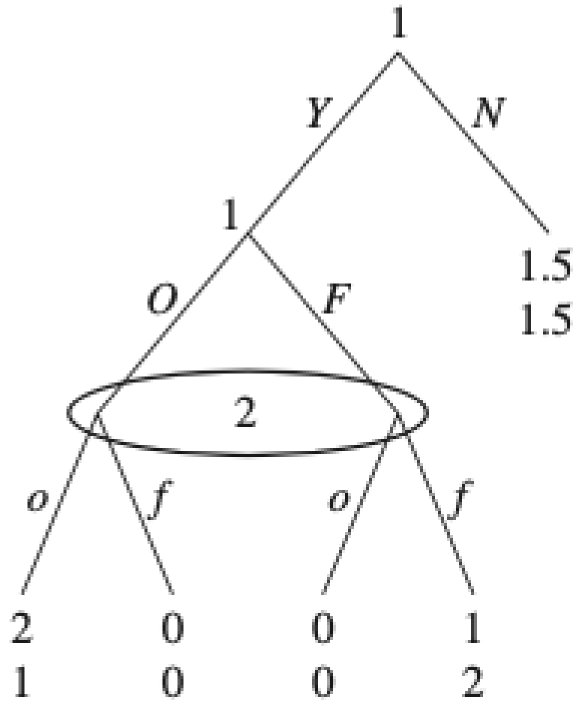

FIGURE 8.2 The voluntary Battle of the Sexes game.

Figure 4: Voluntary battle of sexes game in Tadlis Chapter 8.

Definition 12. A proper subgame G of an extensive-form game Γ consists of only a single node and all its successors in Γ with the property that if $x \in G$ and $x' \in h(x)$ then $x' \in G$. The subgame G is itself a game tree with its information stes and payoffs inherited from Γ.

[2] Recall that we have defined belief in Definition 10, but we haven't talked about belief yet. Indeed, introduce belief can solve this issue in Perfect Bayesian equilibrium, and we will talk later. But now, let's focus on a weaker refinement, which is subgame perfect Nash equilibrium.

In other words, if you want to include a node in your subgame, you should also include all the other nodes in that information set. In addtion, from the definition we know that any subgame must begin with a single node. Below figure 5 is all subgames in the voluntary battle of sexes game

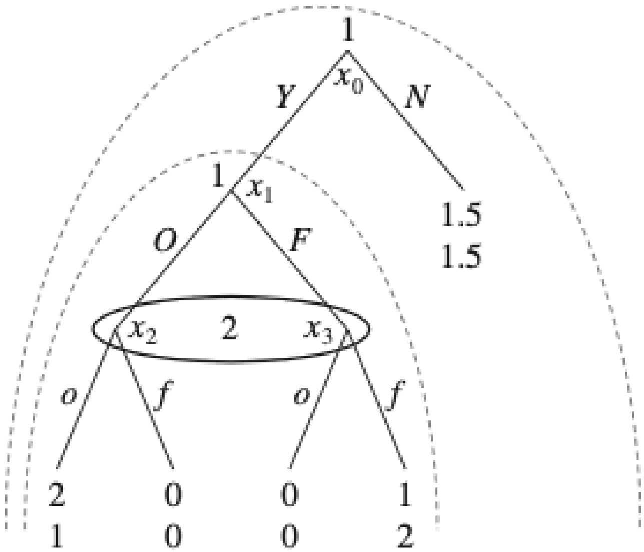

FIGURE 8.4 Proper subgames in the voluntary Battle of the Sexes game.

Figure 5: Proper subgames in voluntary battle of sexes game in Tadlis Chapter 8.

Definition 13. Let Γ be an n-player extensive-form game. A behavioral strategy profile $\sigma^* = (\sigma_2^*, \sigma_2^*, ...\sigma_n^*)$ is a subgame-perfect Nash equilibrium if for every proper subgame G of Σ the restriction of σ^* to G is a Nash equilibrium in G.

Remark. *If we think it carefully, we will realize that backward induction technique in perfect information is the special case of subgame perfect NE. Because in perfect information game, every node is a singleton and hence can be a root of a subgame. And in backward induction, we consider the sequential rationality in each node, this* *will lead the strategy survived in backward induction be a NE. Then strategies profile survived is subgame perfect Nash equilibrium.*

Subgame perfection requires not only that a Nash equilibrium profile of strategies be a combination of best responses on the equilibrium path, which is a necessary condition of a Nash equilibrium, but also that the profile of strategies consist of mutual best responses off the equilibrium path.[3] This is precisely what follows from the requirement that the restriction of the strategy profile σ^* be a Nash equilibrium in every proper subgame, including those subgames that are not reached in equilibrium.

2.5 Perfect Bayesian equilibrium

[4] In last section, we have put forward the subgame perfect NE definition to solve the issue of when backward induction fails due to the uncertainty at information sets in imperfect information game. In other words, we don't consider all nodes' sequential rationality, but only consider the NE in proper subgames.

But we are not satisfied with that. One reason is that some times there is only one subgame which is the whole game, subgame perfect NE doesn't help for refinement. Can we apply sequential rationality for refinement, even in the non-singlton information set? Actually, we can, with the help of belief, we can do more refinement called perfect Bayesian equilibrium.

Definition 14. A **system of beliefs** is a mapping $\mu : x \to [0, 1]$ such that, for all $h_i \in H_i$ and $i \in I$, $\sum_{x \in h_i} \mu(x) = 1$.

Definition 15. A profile of strategies, $\sigma^* = (\sigma_1^*, ..., \sigma_n^*)$, and a system of beliefs, μ, is a **weak perfect Bayesian equilibrium**, if:

1 σ^* is sequentially rational given μ.

[3] Recall the battle of sexes game in figure 2

[4] We skip the discussion about Bayesian Nash equilibrium in static Bayesian game of incomplete information, this is a fundamental of the PBE.

2 μ is derived from σ^* through Bayes rule whenever possible (on the equilibrium path).

Remark. *The "weak" in WPBE is because absolutely no restrictions are being placed on beliefs at information sets that do not occur with positive probability in equilibrium.*

However, the PBE only says we can assign arbitrary belief for the information sets which we will not reach to. However, some will argue that some beliefs in those information sets are not very "reasonable". Can we add some restrictions on beliefs that are off the equilibrium path? Yes, we can do further refinement in those off path strategy. That is **sequential equilibrium**.

2.6 Sequential equilibrium

[5]

Definition 16. $\hat{\sigma}$ is totally mixed if $supp(\hat{\sigma}_{i(h)}(h)) = A(h)$.

In other words, all information sets are reached with positive probability.

Definition 17. Consistency The profile of strategies $\sigma^* = (\sigma_1^*, ..., \sigma_n^*)$, together with a system of beliefs μ^*, is **consistent** if there exists a sequence of totally mixed strategy profiles $\{\sigma^k\}_{k\geq 0}$ and a sequence of beliefs that are derived from each $sigma^k$ according to Bayes' rule, $\{\mu^k\}_{k\geq 0}$, such that $lim_{k\to\infty}(\sigma^k, \mu^k) = (\sigma^*, \mu^*)$.

Definition 18. Sequential equilibrium: A profile of strategies $\sigma^* = (\sigma_1^*, ..., \sigma_n^*)$, together with system of belief μ^*, is a sequential equilibrium if (σ^*, μ^*) is a consistent perfect Bayesian equilibrium.

Theorem 5. A sequential equilibrium exists for every finite extensive-form game.

[5]Reference here

2.7 Intuitive criterion

[6]

Is sequential equilibrium(SE) a good enough refinement? In SE, we carefully define a consistent belief at off-path information sets. This is absolutely a improvement than PBE, which can assign arbitrary beliefs at un-reached information sets. But we will see in this section, a consistent beliefs may still be irrational. A high level summary is that Cho-Kreps criterion rules out Sequential equilibriums that reply on unrealistic beliefs.

Let's consider the signalling game in figure 6[7]:

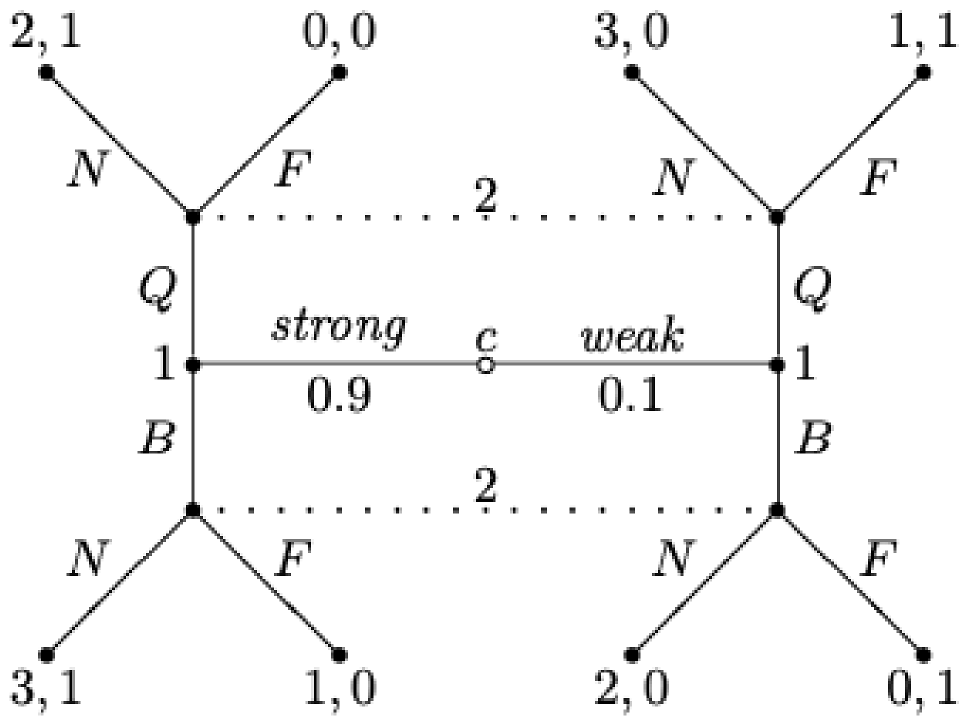

Figure 6: Beer and Quiche: The entry-deterrence problem

A signaling game in which there are two types of player 1, strong and weak, the probabilities of these types are 0.9 and 0.1 respectively,

[6]Reference here

[7]Game is designed here

the set of messages is B, Q (the consumption of beer or quiche for breakfast), and player 2 has two actions, F(ight) or N(ot). Player 1's payoff is the sum of two elements: she obtains two units if player 2 does not fight and one unit if she consumes her preferred breakfast (B if she is strong and Q if she is weak). Player 2's payoff does not depend on player 1's breakfast; it is 1 if he fights the weak type or if he does not fight the strong type. This game has two types of sequential equilibrium, as follows.

- Both types of player 1 choose B, and player 2 fights if he observes Q and not if he observes B. If player 2 observes Q then he assigns probability of at least 0.5 that player 1 is weak.

- Both types of player 1 choose Q, and player 2 fights if he observes B and not if he observes Q. If player 2 observes B then he assigns probability of at least 0.5 that player 1 is weak.

The following argument suggests that an equilibrium of the second type is not reasonable. If player 2 observes that player 1 chose B then he should conclude that player 1 is strong, as follows. If player 1 is weak then she should realize that the choice of B is worse for her than following the equilibrium (in which she obtains the payoff 3), whatever the response of player 2. Further, if player 1 is strong and if player 2 concludes from player 1 choosing B that she is strong and consequently chooses N, then player 1 is indeed better off than she is in the equilibrium (in which she obtains 2). Thus it is reasonable for a strong type of player 1 to deviate from the equilibrium, anticipating that player 2 will reason that indeed she is strong, so that player 2's belief that player 1 is weak with positive probability when she observes B is not reasonable.

Why this unreasonable situation happens? In sequential equilibrium, our consistency condition do not exclude the "stupid" strategy of player 1 with weak type, and we assign positive probability to weak type when observes action B. This is unreasonable, because weak type never chooses B!

To solve this weakness, we introduce the Cho-Kreps **intuitive criterion**.

Definition 19. Two steps for Intuitive criterion:

1 Which type of senders could benefit by deviating from their equilibrium message?

$$\Theta^{**}(m) = \{\theta \in \Theta | u_i^*(\theta) \leq max_{a \in BR(\Theta, m)} u_i(m, a, \theta)\}.$$

Intuitively, we restrict our attention to those types of agents for which sending the off-the-equilibrium message m **could** given them a higher utility than that in equilibrium $u_i^*(\theta)$. If m does not satisfy this inequality, we say that m is "equilibrium dominated".

2 If deviations can only come from the senders identified in the first step, is the lowest payoff from deviating higher than their equilibrium payoff? (Or equivalently, is there a type θ' who always wants to defect with m for all possible beliefs as long as the receiver believes that m is from the type in step 1).

$$min_{a \in BR(\Theta^{**}(m), m)} u_i(m, a, \theta') > u_i^*(\theta), \exists \theta' \in \Theta^{**}(m).$$

If this is true, then we said the equilibrium violates the Intuitive criterion.

Remark. *Possible speech from the sender with incentives to deviate: "It is clear that my type is in $\Theta^{**}(m)$. If my type was outside $\Theta^{**}(m)$ I would have no chance of improving my payoff over what I can obtain at the equilibrium (condition (1)). We can therefore agree that my type is in $\Theta^{**}(m)$. Hence, update your believes as you wish, but restricting my type to be in $\Theta^{**}(m)$. Given these beliefs, any of your best responses to my message improves my payoff over what I would obtain with my equilibrium strategy (condition (2)). For this reason, I am sending you such off-the-equilibrium message."*

Why does IC improves the refinement. It excludes some irrational node which are never happened, this make the probability measure of that information set less than PBEs.

2.8 Strengthened intuitive criterion

Kreps and Cho also defined a stronger intuitive criterion called **Strengthened intuitive criterion**. In intuitive criterion, we said if there is a type of sender who always defects once the receiver knows $\Theta^{**}(m)$ and makes best response on the any beliefs of $\Theta^{**}(m)$.

But sometimes, the intuitive criterion fails because the type in $\Theta^{**}(m)$ will not always defect. We introduce strengthened intuitive criterion:

Definition 20. Also follow two steps:

1 Which type of senders could benefit by deviating from their equilibrium message?

$$\Theta^{**}(m) = \{\theta \in \Theta | u_i^*(\theta) \leq max_{a \in BR(\Theta,m)} u_i(m, a, \theta)\}.$$

2 If deviations can only come from the senders identified in the first step. Does there exist a type $\theta(\mu)$ who always wants to defect with m for all possible beliefs μ as long as the receiver believes that m is from the type in step 1.

$$min_{\{a \in BR(\Theta^{**}(m),m), \theta \in \Theta^{**}(m)\}} u_i(m, a, \theta) > u_i^*(\theta).$$

If this is true, then we said the equilibrium violates the Intuitive criterion.

Remark. *In other words, we loose the* ***rejection conditions*** *in step 2, we doesn't require the defected type always be the same. So some equilibriums survive in intuitive criterion will be deleted furthermore.*

2.9 Divinity criterion)

[8]

In section 2.8, we modify the step 2 of intuitive criterion for refinement. In this section, we will modify step 1 of intuitive criterion for the refinement. This will lead to divinity criterion.

[8] Reference here

Definition 21. Divinity criterion(D1-criterion)
Two steps:

1 Which type of senders are more likely to deviate from their equilibrium message? In particular, for which type of senders are most of the responder's actions beneficial?

Let's first introduce some notation:

$D(\theta, \hat{\Theta}, m) := \cup_{\mu:\mu(\hat{\Theta}|m)=1}\{a \in MBR(\mu, m)|u_i^*(\theta) \leq u_i(m, a, \theta)\}$

$\mathrm{D}^0(\theta, \hat{\Theta}, m) := \cup_{\mu:\mu(\hat{\Theta}|m)=1}\{a \in MBR(\mu, m)|u_i^*(\theta) = u_i(m, a, \theta)\}$

Intuition: $D(\theta, \hat{\Theta}, m)$ is the set of mixed best responses (MBR) of the receiver such that the θ-type of sender is better-off by sending message m than the equilibrium message m^*. [Note that $\mu(\hat{\Theta}|m) = 1$ represents that the receiver believes that message m only comes from types in the subset $\Theta \in \Theta$.]

Now let's identify which type of senders are more likely to deviate from their equilibrium message:

$$[D(\theta, \hat{\Theta}, m) \cup D^0(\theta, \hat{\Theta}, m)] \in D(\theta', \hat{\Theta}, m)$$

That is, for a given message m, under belief that message is from types set $\hat{\Theta}$, and $\theta', \theta \in \hat{\Theta}$. We found θ' is more likely to send message m than θ.

We iterate for all types for all possible $\hat{\Theta}$. The types survived is denoted by $\Theta^{**}(m)$.

2 If deviations can only come from the senders identified in the First Step, is the lowest payoff from deviating higher than their equilibrium payoff?

If the answer is yes, then equilibrium violates the D1-criterion.

We can further restrict step 1 to involve D2-criterion.

Definition 22. D2-criterion:

1 ...

$$[D(\theta, \hat{\Theta}, m) \cup D^0(\theta, \hat{\Theta}, m)] \in \cup_{\theta' \in \{\hat{\Theta} - \theta\}} D(\theta', \hat{\Theta}, m)$$

2 Same with D1 criterion.

Intuitively, in D2 criteria: There are more beliefs (equivalently, MBR) under which some other type strictly wants to defect with m than beliefs under which θ strictly wants to defect or is indifferent towards defection with m.

2.10 Refinement sets

Theorem 6. For general signalling games, we have [9]

Nash equilibrium ⊃ Subgame perfect Nash equilibrium ⊃ Perfect Bayesian equilibrium ⊃ Intuitive criterion ⊃ Strengthened intuitive criterion ⊃ Divinity equlibrium ⊃ D1-criterion ⊃ D2-criterion ⊃ Universal divinity.

[9]Reference here