Glosten-Milgrom (1985) Notes

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1 Overview

This note is to summarize the core idea of paper Glosten, and Milgrom (1985), "Bid, ask and transaction prices in a specialist market with heterogeneously informed traders".

2 Model Setup

- There are two types of traders on the market. Speculators who have private information with probability π, and noise traders who trade with liquidity purpose with probability $1 - \pi$.
- We assume that the market is perfectly competitive, which means that the market maker earns zero expected profit in equilibrium.
- The fundamental value of a stock V is a random variable, but it can be observed by speculators who have insider information about the stock.
- The trade size is allowed as 1 unit and the market maker fills the trade sequentially. This is for simplicity; the insider trader strategy profile is {Buy, Sell, Stay} without considering the size of the trade.
- The equilibrium of this signalling game is summarized below.

  The market maker chooses a pair of bid and ask price $\{a_t, b_t\}$.

  The speculator chooses actions from {Buy, Sell, Stay} based on the insider information he received. We denote the actions as $\{d_t = 1, -1, or\ 0\}$

  s.t

  Market maker earns zero profit and speculator chooses best response (maximizing his profit) in the equilibrium.

3 Solution

- It is trivial that the zero profit condition of market maker can be represented as

$$a_t = E[V_t|\Omega_{t-1}, Buy] \tag{1}$$

$$b_t = E[V_t|\Omega_{t-1}, Sell] \tag{2}$$

where Ω is the information set.

- The profit of speculator is

$$\Pi(V_t, a_t, b_t, d_t) = \begin{cases} V_t - a_t, \ if \ d_t = 1 \\ 0, \ if \ d_t = 0 \\ b_t - V_t, \ if \ d_t = -1 \end{cases} \tag{3}$$

- The best response function of speculator is

$$d_t^*(V_t, a_t, b_t) = \begin{cases} 1, \ if \ V_t > a_t \\ 0, \ if \ b_t > V_t > a_t \\ -1, \ if \ V_t < b_t \end{cases} \tag{4}$$

- When the market maker sees a buying order,

  [1] the market maker knows the order is submitted either by a noise trader with probability $(1-\pi)\beta_b$ where β_b is the probability of noise trader submits a buying order. At this situation, that means no new information is included, so that the expected fundamental value μ_t is

$$\mu_t = E[V_t|\Omega_{t-1}]. \tag{5}$$

  [2] Or the order is submitted from speculators with probability $\pi P(V_t > a_t|\Omega_{t-1})$. At this scenario, the expected fundamental value is update to

$$\mu_t = E[V_t|\Omega_{t-1}, V_t > a_t] \tag{6}$$

- According to equation (1) and Bayesian rule, we can determine the ask price set by market maker,

$$\begin{aligned} a_t &= E[V_t|\Omega_{t-1}, Buy] \\ &= \frac{P(Buy, Noise\ Trader|\Omega_{t-1}, V_t)}{P(Buy|\Omega_{t-1}, V_t)} \times E[V_t|\Omega_{t-1}] \\ &+ \frac{P(Buy, Speculator|\Omega_{t-1}, V_t)}{P(Buy|\Omega_{t-1}, V_t)} \times E[V_t|\Omega_{t-1}, V_t > a_t] \\ &= \frac{(1-\pi)\beta_b}{(1-\pi)\beta_b + \pi P(V_t > a_t|\Omega_{t-1})} E[V_t|\Omega_{t-1}] \\ &+ \frac{\pi P(V_t > a_t|\Omega_{t-1})}{(1-\pi)\beta_b + \pi P(V_t > a_t|\Omega_{t-1})} E[V_t|\Omega_{t-1}, V_t > a_t]. \end{aligned} \tag{7}$$

- Similarly, the bid price is set as,

$$\begin{aligned} b_t &= E[V_t|\Omega_{t-1}, Sell] \\ &= \frac{P(Sell, Noise\ Trader|\Omega_{t-1}, V_t)}{P(Sell|\Omega_{t-1}, V_t)} \times E[V_t|\Omega_{t-1}] \\ &+ \frac{P(Sell, Speculator|\Omega_{t-1}, V_t)}{P(Sell|\Omega_{t-1}, V_t)} \times E[V_t|\Omega_{t-1}, V_t < b_t] \\ &= \frac{(1-\pi)\beta_s}{(1-\pi)\beta_s + \pi P(V_t < b_t|\Omega_{t-1})} E[V_t|\Omega_{t-1}] \\ &+ \frac{\pi P(V_t < b_t|\Omega_{t-1})}{(1-\pi)\beta_s + \pi P(V_t < b_t|\Omega_{t-1})} E[V_t|\Omega_{t-1}, V_t < b_t]. \end{aligned} \tag{8}$$

- The ODE of (7) and (8) is solvable given the conditional probability distribution of $V_t|\Omega_{t-1}$.

4 Exercise

- In previous section, we find the solution of the bid ask price, which replies on the distribution of $V_t|\Omega_{t-1}$.
- In this section, we add some extra assumption to simplify the solution. [Assumption]: The fundamental value is a binary outcome,

$$P(V_t = V^H|\Omega_{t-1}) = \theta, P(V_t = V^L|\Omega_{t-1}) = 1 - \theta \tag{9}$$

- This assumption can simplify expression:

$$E[V_t|\Omega_{t-1}] = \theta V^H + (1-\theta)V^L := \mu \tag{10}$$

- Now we focus on a special equilibrium which satisfies[1]

$$V^L < b_t < a_t < V^H \tag{11}$$

- This equilibrium can further simplify expressions:

$$P(V_t > a_t|\Omega_{t-1}) = \theta \tag{12}$$

$$E[V_t|\Omega_{t-1}, V_t > a_t] = V^H \tag{13}$$

$\Rightarrow$

$$\begin{aligned}
a_t = & \frac{(1-\pi)\beta_b}{(1-\pi)\beta_b + \pi P(V_t > a_t|\Omega_{t-1})} E[V_t|\Omega_{t-1}] \\
& + \frac{\pi P(V_t > a_t|\Omega_{t-1})}{(1-\pi)\beta_b + \pi P(V_t > a_t|\Omega_{t-1})} E[V_t|\Omega_{t-1}, V_t > a_t] \\
& = \frac{(1-\pi)\beta_b}{(1-\pi)\beta_b + \pi\theta}\mu + \frac{\pi\theta}{(1-\pi)\beta_b + \pi\theta} V^H \\
& = \mu + \frac{\theta(1-\theta)\pi}{(1-\pi)\beta_b + \pi\theta}(V^H - V_L)
\end{aligned} \tag{14}$$

$$b_t = \mu - \frac{\theta(1-\theta)\pi}{(1-\pi)\beta_s + \pi(1-\theta)}(V^H - V_L) \tag{15}$$

$$Spread = a_t - b_t = \frac{[(1-\pi)(\beta_s + \beta_b) + \pi]\theta(1-\theta)\pi(V^H - V^L)}{[(1-\pi)\beta_b + \pi\theta][(1-\pi)\beta_s + \pi(1-\theta)]} \tag{16}$$

[1]This suppose will be verified at equation (14) and (15).