Grossman (1976) Notes

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1 Overview

This note is to summarize the core idea of paper Grossman (1976), "On the Efficiency of Competitive Stock Markets Where Trades Have Diverse Information".

2 Model Setup

- This is a single period model. There are two types of assets.

  The risk-free asset gives out R_f at time 1 and costs \$1 at time 0.

  The risky asset costs p_0 at time 0, and the payoff at time 1 is a random variable P_1.

- The unconditional distribution of the payoff P_1 follows normal distribution $\sim N(\mu, \sigma^2)$.

- The total supply of risk asset is fixed as $\bar{X}$

- There are n different traders. The $i-th$ trader has begining wealth $W_{0,i}$.

- Traders have CARA utility functions

$$U_i(W_{1,i}) = -exp\{-a_i W_{1,i}\} \tag{1}$$

- Traders can observe some private signals for themselves, the signals include the real payoff adding some noise,

$$Y_i = P_1 + \epsilon_i, \tag{2}$$

  where $\epsilon_i \sim N(0, \sigma_i^2)$.

3 Solution

- If trader i buys X_i at beginning, then her payoff at time 1 is

$$W_{1,i} = R_f W_{0,i} + [P_1 - R_f p_0] X_i. \tag{3}$$

- The conditional expectation and variance of $W_{1,i}|I_i$ are

$$E[W_{1,i}|I_i] = X_i[E[P_1|I_i] - R_f p_0] \tag{4}$$

$$V[W_{1,i}|I_i] = X_i^2 V[P_1|I_i] \tag{5}$$

- We denote I_i as the information available to $i-th$ trader. The trader's obejective function can be represented as

$$\begin{aligned}\max_{X_i} E[U_i(W_{1,i})|I_i] &= \max_{X_i} E[-exp\{-a_i W_{1,i}\}|I_i] \\ &= \max_{X_i} a_i(E[W_{1,i}|I_i] - \frac{1}{2} a_i V[W_{1,i}|I_i])\end{aligned} \tag{6}$$

$$\begin{aligned}&\Rightarrow \max_{X_i} E[W_{1,i}|I_i] - \frac{1}{2} a_i V[W_{1,i}|I_i] \\ &= \max_{X_i} X_i[E[P_1|I_i] - R_f p_0] - \frac{1}{2} a_i X_i^2 V[P_1|I_i].\end{aligned} \tag{7}$$

$$F.O.C \Rightarrow X_i^* = \frac{E[P_1|I_i] - R_f p_0}{a_i V[P_1|I_i]}. \tag{8}$$

- Considering that trader only uses her private information which means $I_i = \{Y_i\}$, the conditional value of equation (4), (5), and (8) can be rewritten as

$$E[P_1|Y_i] = \mu + \rho_i^2 (Y_i - \mu)^1 \tag{9}$$

$$V[P_1|Y_i] = \sigma^2 (1 - \rho_i^2), \tag{10}$$

where $\rho_i = \frac{\sigma}{\sqrt{\sigma^2 + \sigma_i^2}}$ is the correlation between Y_i and P_1.

$$\Rightarrow X_i^* = \frac{\mu + \rho_i^2 (Y_i - \mu) - R_f p_0}{a_i \sigma^2 (1 - \rho_i^2)}.^2 \tag{11}$$

[1]Reference: `https://online.stat.psu.edu/stat414/lesson/21/21.1`

[2]It shows demand is increasing in the precision of the signal (the closer is ρ_i to 1, that is, the lower is σ_i).

- The market clearing condition implies

$$\bar{X} = \sum_{i=1}^{n} X_i^* \tag{12}$$

$$\Rightarrow p_0 = \frac{1}{R_f}[\sum_{i=1}^{n} \frac{\mu + \rho_i^2(Y_i - \mu)}{a_i\sigma^2(1-\rho_i^2)} - \bar{X}]/[\sum_{i=1}^{n} \frac{1}{a_i\sigma^2(1-\rho_i^2)}] \tag{13}$$

- However, it is not fully rational equilibrium price because the solution neglects that traders might learn about other traders' signals from p_0 itself (price discovery).

- In other words, the information set for $i - th$ trader should be $I_i = \{Y_i, p_0^*\}$, where p_0^* is the rational expectations equilibrium price.

- For simplicity, we further assume that $\sigma_i^2 = \sigma_\epsilon^2$ and independent for $i = 1, 2, ..., n$.

4 Rational Expectations Equilibrium Price

In this section, we will prove there exists a rational expectations equilibrium with p_0^* given by

$$p_0^* = \frac{1-\rho^2}{R_f}\mu + \frac{\rho^2}{R_f}\bar{Y} - \frac{\sigma^2(1-\rho^2)}{R_f\sum_{i=1}^{n}\frac{1}{a_i}}\bar{X}, \tag{14}$$

where $\bar{Y} := \frac{1}{n}\sum_{i=1}^{n} Y_i$ and $\rho^2 := \frac{\sigma^2}{\sigma^2 + \frac{\sigma_\epsilon^2}{n}}$. ρ is the correlation between P_1 and $\bar{Y}$.

$\boxed{Proof:}$

- Given the observable price p_0^*, the average signal $\bar{Y}$ can be inferred for all traders.

- Equation (8) can be rewritted as,

$$X_i^* = \frac{E[P_1|Y_i, \bar{Y}] - R_f p_0^*}{a_i V[P_1|Y_i, \bar{Y}]} \tag{15}$$

- Because $\bar{Y}$ is a stronger signal compaered with private signal Y_i, which makes Y_i redundant. Such that

$$E[P_1|Y_i, \bar{Y}] = E[P_1|\bar{Y}] = \mu + \rho^2(\bar{Y} - \mu) \tag{16}$$

$$V[P_1|Y_i, \bar{Y}] = V[P_1|\bar{Y}] = \sigma^2(1-\rho^2). \tag{17}$$

$$\Rightarrow X_i^* = \frac{\mu + \rho^2(\bar{Y} - \mu) - R_f p_0^*}{a_i \sigma^2 (1-\rho^2)} \tag{18}$$

- Plug equation (14) into equation (18)

$$\Rightarrow \sum_{i=1}^{n} X_i^* = \bar{X} \tag{19}$$

- Market clearing condition is also satisfied, we have proved this equilibrium exists.

5 Robustness of Results

- Note that the equilibrium breaks down if each trader i needed to pay a tiny cost, c, to obtain Y_i since there is not personal benefit from Y_i (Grossman and Stiglitz, 1980). There is free ride.

- This is also known as the Grossman-Stiglitz Paradox that argues perfectly informationally efficient markets are an impossibility since, if prices perfectly reflected available information, there is no profit to gathering information, in which case there would be little reason to trade and markets would eventually collapse.

6 Reference

- Grossman, S., 1976. On the efficiency of competitive stock markets where trades have diverse information. The Journal of finance, 31(2), pp.573-585.

- `https://gpennacc.web.illinois.edu/ChPres16.pdf`

- `https://markus.scholar.princeton.edu/sites/g/files/toruqf2651/files/03_ree.pdf`