Hellwig(1980), and Grossman-Stiglitz(1980) Notes: Continuous CARA-Gaussian Model with Informed, Uniformed and Noise Traders

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December 31, 2024

1 Overview

This note is to summarize the core idea of paper Hellwig(1980), "On the aggregation of information in competitive markets", and Grossman-Stiglitz paradox. This material includes the slides from `https://blog.iese.edu/xvives/files/2011/09/ch4-slides-compact.pdf` by Vives.

2 Model Setup

- There is a single risky asset with random value θ follows

$$\theta \sim N(\bar{\theta}, \sigma_\theta^2) \tag{1}$$

- Risk averse agents in the uniform interval $[0, 1]$ and noise traders.

  1. A fraction of traders $\mu \in [0, 1]$ are informed traders who receive a private signal s_i about *theta*, and the signal is a random variable,

  $$s_i = \theta + \epsilon_i, \tag{2}$$

  where $\epsilon_i \sim N(0, \sigma_\epsilon^2)$. In other words, every informed traders' signals have same precision.

  2. Noise trader's submit random trade u follows

  $$u \sim N(0, \sigma_u^2) \tag{3}$$

- The utility derived by a trader i for the profit $\pi_i = (\theta - p)x_i$ of buying x_i units of the asset at price p is of the CARA type,

$$U(\pi_i) = -exp\{\rho_i \pi_i\}, \tag{4}$$

where $\rho_i = \rho_I > 0, \forall i \in [0, \mu]$, and $\rho_i = \rho_U > 0, \forall i \in (\mu, 1]$.

- By convention, given θ as the average signal of a positive mass μ of agents,

$$\frac{1}{\mu}\int_0^\mu s_i di = \theta, almost\ surely \tag{5}$$

3 Solution

- We look for symmetric rational expectations equilibrium, which is defined as: $\{X_I(s_i, p)\ for\ i \in [0, \mu]; X_U(p)\ for\ j \in (\mu, 1]\}$, and the price function $P(\theta, u)$ s.t

$$\text{Market clearing} \leftrightarrow \int_0^\mu X_I(s_i, p)di + \int_\mu^1 X_U(p)dj + u = 0,\ almost\ surely. \tag{6}$$

$$X_I(s_i, p) \in \arg\max_x E[U_i((\theta - p)x)|s_i, p] \tag{7}$$

$$X_I(p) \in \arg\max_x E[U_j((\theta - p)x)|, p] \tag{8}$$

for $i \in [0, \mu], j \in (\mu, 1]$.

- Additionally, we focus a special linear equilibrium where $p(\theta, u)$ is linear in θ and u. This assumption leads $(\theta - p)$ also follows normal distribution.

- Given the linearity of price, and denote $G = \{s_i, p\}(G = \{p\})$ for the informed(uniformed) trader. The trader i's expected utility given strategy x_i is

$$\begin{aligned} E[U(\pi_i)|G] &= E[-exp\{-\rho_i \pi_i\}|G] \\ &= -exp\{\rho_i(E[\pi_i|G] - \frac{\rho_i}{2}V[\pi_i|G])\} \\ &= -exp\{\rho_i(E[\theta - p|G]x_i - \frac{\rho_i}{2}V[\theta - p|G]x_i^2)\}. \end{aligned} \tag{9}$$

$$F.O.C \Rightarrow x_i = \frac{E[\theta|G] - p}{\rho_i V[\theta|G]} \text{ is the Bayesian Nash equilibrium.} \tag{10}$$

4 Linear Rational Expectations Equilibrium Price

Proposition 1 *Given the linearity assumption of price. And let* $\rho_I > 0$ *and* $\rho_U > 0$. *There is a unique Bayesian linear equilibrium in demand functions. It is given by:*

$$X_I(s_i, p) = a(s_i - p) - b_I(p - \bar{\theta}) \tag{11}$$

$$X_U(p) = -b_U(p - \bar{\theta}), \tag{12}$$

where $a = (\rho_I \sigma_\epsilon^2)^{-1}$, *and*

$$b_I = \frac{1/\sigma_\theta^2}{\rho_I + \mu(1/\sigma_\epsilon^2)(1/\sigma_u^2)(\mu\rho_I^{-1} + (1-\mu)\rho_U^{-1})} \tag{13}$$

$$b_U = \frac{\rho_I}{\rho_U} b_I \tag{14}$$

In addition, p *is derived as*[1]

$$p = \bar{\theta} + \frac{1}{\mu(a + b_I) + (1-\mu)b_U}(\mu a(\theta - \bar{\theta}) + u) \tag{15}$$

The proof is shown in the slide, the sketch is following below steps:

- Conjecture equilibrium strategies of the form

$$X_I(s_i, p) = as_i - c_I p + \hat{b_I} \tag{16}$$

$$X_U(p) = -c_U p + \hat{b_U} \tag{17}$$

- Impose market clearing to get the price, which equals to

$$p = \frac{\mu a\theta + u + \mu\hat{b_I} + (1-\mu)\hat{b_U}}{\mu c_I + (1-\mu)c_U} \tag{18}$$

- Combine (16), (17), (18), and (10). We can get $a, c_I, \hat{b_I}, c_U, \hat{b_U}$
- $\Rightarrow$ (11), (12), and (15) are derived.

[1] Note: it still satisfies the linearity assumption in θ and u.

5 Discussion

- Unlike the model in Grossman(1976), in equation (11), we find that the strategies of informed traders depend on not only on the price, but also on their own signals. That means the price is not fully informational. Informed trader can get benefit from the private signal he received. This is due to the noise trader who interfere uninformed trader learning from informed trader.
- $\lambda = \frac{1}{\mu(a+b_I)+(1-\mu)b_U}$ is the depth of the market, that is, the change in price due to a 1 unit change in the demand of noise traders.
- Following the Vives' note. After some special setting, e.g.
  1. Signal is same for all informed traders.
  2. Observation of signal costs k.
  3. No noise trader, $\sigma_u \to 0$.
  4. ...
- Under this settings, there is no equilibrium (Grossman-Stiglitz(1980)): In the absence of noise, no one has an incentive to acquire private information (It is costly can will be learned from uninformed trader, this is known as Grossman-Stiglitz paradox.

6 Reference

- Hellwig, M.F., 1980. On the aggregation of information in competitive markets. Journal of economic theory, 22(3), pp.477-498.
- Grossman, S.J. and Stiglitz, J.E., 1980. On the impossibility of informationally efficient markets. The American economic review, 70(3), pp.393-408.
- `https://blog.iese.edu/xvives/files/2011/09/ch4-slides-compact.pdf`