Investment Based CAPM: q-Factor Model

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1 Overview

In this note, we introduce the q-factor model by Hou, Xue, and Zhang (2015). They derive a 4-factor model from the perspective of investment choise of company.

2 q-Factor Model

- Assume a company is facing a two-period investment choice. At time 0, company i's asset is $A_{0,i}$, today's profitability is $\Pi_{i,0}$.
- The profitability in date 1 is a random variable $\Pi_{i,1}$.
- Company decides how much to investment today $I_{i,0}$. We further assume that the asset will be completely depreciated at time 1. In other words, we have
$$A_{i,1} = I_{i,0}. \tag{1}$$
- Adjustment cost is measured by
$$\frac{a}{2}\frac{I_{i,0}^2}{A_{i,0}} \tag{2}$$
- The cash flow for shareholders at time $t = 0$ is
$$\Pi_{i,0}A_{i,0} - I_{i,0} - \frac{a}{2}\frac{I_{i,0}^2}{A_{i,0}}. \tag{3}$$
- The expected cash flow discounted by stochastic discount factor is
$$E[m\Pi_{i,1}I_{i,0}] \tag{4}$$

- The objective function is,

$$\max_{I_{i,0}} \Pi_{i,0}A_{i,0} - I_{i,0} - \frac{a}{2}\frac{I_{i,0}^2}{A_{i,0}} + E[m\Pi_{i,1}I_{i,0}]. \tag{5}$$

- The F.O.C indicates

$$1 + \frac{a}{A_{i,0}}I_{i,0} = E[m\Pi_{i,1}], \tag{6}$$

- which can be explained as the marginal cost of investment (left hand side) equals to the marginal expected return dicounted to $t = 0$ (right hand side).
- The stock return can be written as,

$$R_{i,1} = \frac{P_{i,1} + D_{i,1}}{P_{i,0}} = \frac{0 + \Pi_{i,1}I_{i,0}}{E[m\Pi_{i,1}I_{i,0}]} = \frac{\Pi_{i,1}}{E[m\Pi_{i,1}]} = \frac{\Pi_{i,1}}{1 + \frac{a}{A_{i,0}}I_{i,0}}. \tag{7}$$

$$\Rightarrow E[R_{i,1}] = \frac{E[\Pi_{i,1}]}{1 + \frac{a}{A_{i,0}}I_{i,0}}. \tag{8}$$

- When expected profitability $E[\Pi_{i,1}]$ is given, stock expected return has negative correlation with investment $I_{i,0}$ today.
- When investment today is given, stock expected return has positive correlation with expected profitability.
- This is the motivation of adding additional investment, and profitability factor to their model.

3 Discussion

- The authors expand the model to multiple-period scenario (q^5 model), the new model says

$$R_{i,t+1} = \frac{\Pi_{i,t+1} + \frac{a}{2}(I_{i,t+1}/A_{i,t+1})^2 + (1-\delta)[1 + a(I_{i,t+1}/A_{i,t+1})]}{1 + a(I_{i,t}/A_{i,t})}. \tag{9}$$

- If we set depreciation rate δ to 0, and in 2-period mode $I_{i,1} = 0$. The model is exactly the same with the one in section 2.

- After taking expectation on both sides,

$$E[R_{i,t+1}] = \frac{E[\Pi_{i,t+1}] + E[\frac{a}{2}(I_{i,t+1}/A_{i,t+1})^2] + E[(1-\delta)[1 + a(I_{i,t+1}/A_{i,t+1})]]}{1 + a(I_{i,t}/A_{i,t})}. \tag{10}$$

- In the numerator, the first term has the the same explanation (expected profitability), the second term can be ignored due to square. The third term is explained as the expected investment to asset growth.

- That is the motivation of adding growth factor in their q^5 model.

4 Reference

- Hou, K., C. Xue, L. Zhang (2015). Digesting anomalies: an investment approach. The Review of Financial Studies, Vol. 28(3), 650 – 705.

- https://www.youtube.com/watch?v=-nRH0ACfGKE&t=586s

- https://www.zhihu.com/people/liu-ru-ming-8