Links Between Stochastic Discount Factor, CAPM, CCAPM, and Black-Scholes

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1 Overview

In asset pricing models, there are typically two primary approaches to determining an asset's price: Arbitrage Pricing Theory (APT) and the equilibrium framework.

APT relies on a replication strategy to perfectly hedge the asset. By constructing a portfolio that replicates the asset's payoffs, the one-price theorem is applied, ensuring that no arbitrage opportunities exist. This process determines the asset's price.

On the other hand, the equilibrium framework takes a more fundamental approach. It begins with individuals' utility functions, using optimization techniques to solve for maximum utility. The market-clearing condition is then applied, balancing demand and supply to establish the equilibrium price of the asset.

However, the stochastic discount factor (SDF) approach based on Euler equation fill the gap between these two approaches, they are equivalent and are specific applications under some appropriate assumptions in SDF approach.

2 Stochastic Discount Factor

- In Lucas tree (1978) model, asset is priced as the present value of future payoff:
$$p_t = E_t^P[m_T \cdot X_T] \tag{1}$$
- The payoff $X(\omega)$ and $m(\omega)$ are both random (stochastic) variable. That's why $m(\omega)$ is called SDF or pricing kernel in asset pricing. P indicates the physical worlds probability.

- In Arrow-Debreu market, $m(\omega)$ is the state price of Arrow-Debreu security which pays 1 when ω_i happens.

3 SDF Approach to CAPM and CCAPM

- Trivially, assume there is an asset which pays random variable $1+R(\omega)$ at time $T=T$ if buying it with price 1 at time $T=t$. We know that

$$1 = E_t^P[m_T \cdot (1+R)] \tag{2}$$

- Additionally, if we select the SDF used in Lucas tree (1978), where C is the consumption random variable.

$$m_T = \frac{u'(C_T)}{u'(C_t)}. \tag{3}$$

- Then we can rewrite equation (2) as,

$$1 = E_t^P[\frac{u'(C_T)}{u'(C_t)} \cdot (1+R)] = E_t^P[\frac{u'(C_T)}{u'(C_t)}](1+E[R]) + Cov_t^P(\frac{u'(C_T)}{u'(C_t)}, R) \tag{4}$$

- Specifically, we choose a specific asset, the risk free asset, then the equation (4) is wirtten as,

$$\frac{1}{1+r_f} = E_t^P[\frac{u'(C_T)}{u'(C_t)}]. \tag{5}$$

- Now we can plug it into equation (4) and get,

$$1 = \frac{1+E[R]}{1+r_f} + Cov_t^P(\frac{u'(C_T)}{u'(C_t)}, R) \tag{6}$$

- Applying Stein's Lemma[1]

$$1 = \frac{1+E[R]}{1+r_f} + \frac{E_t^P[u''(C_T)]Cov_t^P(C_T, R)}{u'(C_t)}. \tag{7}$$

$$\Rightarrow E[R] - r_f = -\frac{E_t^P[u''(C_T)]Cov_t^P(C_T, R)}{u'(C_t)}(1+r_f) \tag{8}$$

[1] https://en.wikipedia.org/wiki/Stein%27s_lemma

- Specifically, if the asset is the market portfolio M, then we have

$$\Rightarrow E[R_M] - r_f = -\frac{E_t^P[u''(C_T)]Cov_t^P(C_T, R_M)}{u'(C_t)}(1 + r_f) \tag{9}$$

$$\Rightarrow \frac{E[R] - r_f}{Cov_t^P(C_T, R)} = \frac{E[R_M] - r_f}{Cov_t^P(C_T, R_M)} = Const = \frac{E_t^P[u''(C_T)]}{u'(C_t)}{}^{2} \tag{10}$$

- The equation (10) can be re-arrange to CCAPM:

$$E[R] - r_f = (E[R_M] - r_f)\frac{Cov(C, R)}{Cov(C, R_M)}{}^{3} \tag{11}$$

- If we further assume the consumption C only comes from market portfolio, then we have CAPM:

$$E[R] - r_f = (E[R_M] - r_f)\frac{Cov(R_M, R)}{\sigma^2(R_M)}{}^{4} \tag{12}$$

4 SDF Approach to Black Scholes Model

- We derive the Black Scholes model by using SDF approach in continuous time framework. The logic is using the prices of risk-free bank account and risky stock to determine the dynamic of stochastic discount factor. Then use the SDF to calculate the derivative's price.

- The bank account dynamic is,

$$dB(t) = rB(t)dt \tag{13}$$

- The stock price dynamic under physical probability measure is,

$$dS(t) = \mu S(t)dt + \sigma S(t)dW(t) \tag{14}$$

- Assume the dynamic of SDF follows:

$$dM(t) = aM(t)dt + bM(t)dW(t) \tag{15}$$

[2] Also known as Arrow Pratt Measure of Risk Aversion.
[3] Also known as consumption Beta.
[4] Also known as CAPM market Beta.

- According to Bjork (10.48)[5], price of any asset can be rewritten as:

$$\begin{aligned} M(t)S(t) &= E_t^P[M(t+dt)S(t+dt)] \\ &= E_t^P[(M(t)+dM(t))(S(t)+dS(t))] \\ &= E_t^P[M(t)S(t)+S(t)dM(t)+M(t)dS(t)+dM(t)dS(t)] \end{aligned} \tag{16}$$

- Both sides are divided by $M(t)S(t)$, we then have,

$$0 = E_t^P[\frac{dM(t)}{M(t)} + \frac{dS(t)}{S(t)} + \frac{dM(t)}{M(t)} \cdot \frac{dS(t)}{S(t)}]. \tag{17}$$

- If we replace risk asset $S(t)$ to risk-free bank account $B(t)$, we will have,

$$0 = E_t^P[rdt + adt + bdW(t)] \Rightarrow a = -r. \tag{18}$$

- So that we have $dM(t) = -rM(t)dt + bM(t)dW(t)$, equation (17) is rewritten as,

$$0 = E_t^P[-rdt + bdW(t) + \mu dt + \sigma dW(t) + \sigma bdt] \tag{19}$$

$$\Rightarrow b = \frac{r-\mu}{\sigma} \tag{20}$$

$$\Rightarrow dM(t) = -rM(t)dt - \frac{\mu - r}{\sigma}M(t)dW(t) \tag{21}$$

- Similar technique if we replace the risk asset $S(t)$ in Equation (17) with the European call option $C(t, S)$

$$0 = E_t^P[\frac{dM(t)}{M(t)} + \frac{dC(t,S(t))}{C(t,S(t))} + \frac{dM(t)}{M(t)} \cdot \frac{dC(t,S(t))}{c(t,S(t))}]. \tag{22}$$

- Applying Ito lemma,

$$dC(t,S(t)) = (C_t + \mu SC_s + \frac{\sigma^2 S^2}{2}C_{ss})dt + \sigma SC_s dW(t). \tag{23}$$

- Equation (17) can be transformed to

$$0 = E_t^P[-rdt - \frac{\mu - r}{\sigma}dW(t) + \frac{C_t + \mu SC_s + \frac{\sigma^2 S^2}{2}C_{ss}}{C}dt + \frac{\sigma SC_s}{C}dW(t) - \frac{\mu - r}{\sigma}\frac{\sigma SC_s}{C}dt] \tag{24}$$

$$\Rightarrow 0 = E_t^p[(-r + \frac{C_t + \mu SC_s + \frac{\sigma^2 S^2}{2}C_{ss}}{C} - \frac{(\mu - r)SC_s}{C})dt + (.)dW(t)] \tag{25}$$

[5]Which says $M \cdot S$ is a local P-martingale.

- dt term must be zero:

$$0 = -rC + C_t + rSC_s + \frac{\sigma^2 S^2}{2} C_{ss} \tag{26}$$

- Which yields to Black-Scholes PDE.

5 Reference

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