INTEREST RATE TERM STRUCTURE NOTES

Ruming Liu

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Contents

1 ¶ Interest Rate

1.1 § Zero Coupon Bond & Interest Rate

Definition 1.1.1. *A zero coupon bond with maturity date T is called T-bond, which is a contract with guarantee paying $1 at time T. The price of a T-bond at time t is denoted by p(t,T).*

Apparently, (1) $p(T,T) = 1$. (2) $p(t,T)$ can be observed at time t in the market, so $p(t,T)$ should be $\mathcal{F}_t$ measurable for all T. (3) $p(0,T)$ and $p(T,T)$ is deterministic, but $p(t,T)$ is random.

We also should have assumption: For each fixed time t, the bond price p(t, T) is differentiable w.r.t time T.

According to Arbitrage Free theorem, at time t, we can get the *Forward Rate* between time [s, T], which is $p(t,S)/p(t,T)$. Based on these materials, we can define some other rates:

(a) LIBOR forward Rate for [S, T] contracted at t:

$$L(t;S,T) = \frac{p(t,S) - p(t,T)}{(T-S)p(t,T)}$$

(b) LIBOR spot rate at t:

$$L(t,T) = L(t;t,T) = \frac{1 - p(t,T)}{(T-t)p(t,T)}$$

(c) Continuously compounded forward rate for [S, T] contracted at t:

$$R(t;S,T) = \frac{logp(t,S) - logp(t,T)}{T-S}$$

(d) Continuously compounded spot rate at t:

$$R(t,T) = \frac{-logp(t,T)}{T-t}$$

(e) Instantaneous forward rate with maturity T, contracted at t:

$$f(t,T) = \lim_{S \to T} R(t;S,T) = -\partial_T logp(t,T)$$

(f) Short rate at t is:

$$r(t) = f(t,t) = -\partial_T p(t,t)$$

Definition 1.1.2. *The money account process is defined by:*

$$B_t = exp\{\int_0^t r(s)ds\}$$

This definition is reasonable, because B_t is equivalent to investing in a self--finance "rolling over" strategy, which at each time t consists entirely of "just maturing" bonds:

$$B_t \approx \frac{1}{p(0,\delta)}\frac{1}{p(\delta,2\delta)}... \approx \frac{p(0,0)}{p(0,\delta)}\frac{p(\delta,\delta)}{p(\delta,2\delta)} \approx e^{r_o\delta + r_\delta\delta} = e^{\int_0^t r_s ds}$$

Hence we have Lemma:

Lemma 1.1.1. $p(t,T) = exp\{-\int_t^T f(t,s)ds\}$ *for all* $t < T$.

1.2 § Relations between $f(t,T)$, $p(t,T)$ & $r(t)$

We have defined some interest rates above. Furthermore, we can model these rates and bond price as the following dynamics:

$$dr(t) = a(t)dt + b(t)dW_t \tag{1.1}$$

$$dp(t,T) = p(t,T)m(t,T)dt + p(t,T)v(t,T)dW_t \tag{1.2}$$

$$df(t,T) = \alpha(t,T)dt + \sigma(t,T)dW_t \tag{1.3}$$

Our goal is to find the internal relation between these interest rates or bond price. Before doing that, we also need some technical assumptions:

Assumption 1.2.1.

(1) For each fixed time t, all the objects $m(t,T)$, $v(t,T)$, $\alpha(t,T)$ *and* $\sigma(t,T)$ *are assumed to be continuously differentiable in the T-variable.*

(2) All processes are assumed to be regular enough to allow us to differentiate under the integral sign as well as to interchange the order of integration.

Under these assumptions, the results below hold, regardless of the measure under consideration, and in particular we do **not** assume that markets are free of arbitrage (Those are the topics we will talk later).

Proposition 1.2.1.

1. *If* $p(t,T)$ *satisfies (1.2), then for the forward rate dynamics we have*

$$df(t,T) = \alpha(t,T)dt + \sigma(t,T)dW_t$$

where α *and* σ *are given by*

$$\begin{cases} \alpha(t,T) = v_T(t,T)v(t,T) - m_T(t,T) \\ \sigma(t,T) = -v_T(t,T) \end{cases} \tag{1.4}$$

2. *If* $f(t,T)$ *satisfies (1.3) then the short rate satisfies*

$$dr(t) = a(t)dt + b(t)dW_t$$

where

$$\begin{cases} a(t) = f_T(t,t) + \alpha(t,t) \\ b(t) = \sigma(t,t) \end{cases} \tag{1.5}$$

3. If $f(t,T)$ satisfies (1.3) then $p(t.T)$ satisfies

$$dp(t,T) = p(t,T)\{r(t) + A(t,T) + \frac{1}{2}||S(t,T)||^2\}dt + p(t,T)S(t,T)dW_t$$

where $|| * ||$ denotes the Euclidean norm, and

$$\begin{cases} A(t,T) = -\int_t^T \alpha(t,s)ds \\ S(t,T) = -\int_t^T \sigma(t,s)ds \end{cases} \tag{1.6}$$

Proof. Recall that $f(t,T) = R(t;T,t) = -\partial_T logp(t,T)$ and $r(t) = f(t,t)$, so we will have:

$$\int_t^T f(t,s)ds = -[\ln p(t,T) - \ln p(t,t)] = -\ln p(t,T)$$

$$\Rightarrow p(t,T) = exp\{-\int_t^T f(t,s)ds\}$$

And by Ito Formula, we will have:

$$dp(t,T) = p(t,T)[r(t) - \int_t^T \alpha(t,s)ds + \{\frac{1}{2}\int_t^T \sigma(t,s)ds]^2\}dt - p(t,T)\int_t^T \sigma(t,s)dsdW_t$$

combining this with (1.2) we can get the system (1.4) & (1.6). The proof of (1.5) is shown in book Page-307. □

We can use the diagram to show the relations between these 3 important factors:

$$\boxed{p(t,T) \Leftrightarrow f(t,T) \Rightarrow r(t)}$$

1.3 § Coupon Bonds, Swaps & Yields

We now go to compute the price of the fixed coupon bond, it is obvious that the coupon bond can be replicated by a series of zero coupon bonds. More preciseky we will hold c_i zero coupon bonds of maturity T_i, $i = 1, ..., n-1$, and $K + c_n$ bonds with maturity T_n, so the price, $p(t)$ at time $t < T_1$ is given by:

$$p(t) = Kp(t, T_n) + \sum_{i=1}^{n} c_i p(t, T_i) \tag{1.7}$$

Very often the coupons are determined in terms of **return**, rather than in monetary terms. The return for the ith coupon is typically quoted as a simple rate acting on the face value K, over the period $[T_{i-1}, T_i]$. Thus, if, for example, the ith coupon has a return equal to r_i, which means:

$$c_i = r_i(T_i - T_{i-1}K)$$

For a standardized coupon bond, the time intervals will be equally spaced, i.e.

$$T_i = T_0 + i\delta,$$

and the coupon rates $r_1, ..., r_n$ will be equal to a common coupon rate r. The price $p(t)$ of such a bond for $t \leq T_1$, will be given by

$$p(t) = K[p(t, T_n) + r\delta \sum_{i=1}^{n} p(t, T_i)] \tag{1.8}$$

But sometimes, the coupon is not fixed at the time the bond is issued, we call it **Floating Rate Bonds**. For instance, the coupon rate r_i is set to the spot **LIBOR** rate $L(T_{i-1}, T_i)$ which can be observed at time T_{i-1}. Thus

$$c_i = (T_i - T_{i-1})L(T_{i-1}, T_i)K,$$

now we go on to compute the value of this bond at time $t < T_0$ and assume $T_i - T_{i-1} = \delta$. Without loss of generality we may also assume that $K = 1$, combining the definition of **LIBOR** we have

$$c_i = \delta \frac{1 - p(T_{i-1}, T_i)}{\delta p(T_{i-1}, T_i)} = \frac{1}{p(T_{i-1}, T_i)} - 1.$$

We are going to calculate the present value of coupon c_i at time $t < T_0$, to in case of arbitrage, it must hold:

$$c_i = p(t, T_{i-1}) - p(t, T_i).$$

Summing up all the terms we finally obtain the following valuation formula for the floating rate bond:

$$p(t) = p(t, T_n) + \sum_{i=1}^{n} [p(t, T_{i-1}) - p(t, T_i)] = p(t, T_0) \tag{1.9}$$

In particular we see that if $t = T_0$, then $p(T_0) = 1$. But it is not surprising, because if you can hedge this contract just by putting all the money you have in zero coupon bond whose maturity is T_{i+1} at T_i for $i = 0, ..., n-1$.

I will skip the concepts of Swaps Duration and Convexity.

2 ¶ Short Rate Models

2.1 § Generalities

In this section, we will talk about the relation between short rate $r(t)$ and bond price $p(t, T)$. Intuitively speaking, the price $p(t, T)$ should depends on the behavior of the short rate over the time $[t, T]$. But it is $\boxed{\textbf{not true}}$ according the relations we talked in Proposition 1.2.1, which states that given the dynamic of short rate, the bond price $p(t, T)$ is not determined by the rate. We will discuss why our intuition is wrong later.

First, we model the short rate, under the objective probability measure P, as the solution of an SDE of the form

$$dr(t) = \mu(t, r(t))dt + \sigma(t, r(t))dW_t. \tag{2.1}$$

And we also model the money account by the dynamic

$$dB(t) = r(t)B(t)dt. \tag{2.2}$$

We also assume the market is large, there exists bonds with all possible maturities.

Assumption 2.1.1. *We assume that there exists a market for zero coupon T-bonds for every value of T*

Consequently, it is a market containing infinite assets, but only the exogenously given. In other words, in this model the risk free asset is considered as the underlying asset whereas all bonds are regarded as derivatives of the 'underlying' short rate r. Like before, we want to find the arbitrage free bond prices with different maturities. A natural question is whether the bond prices are uniquely determined by the given r dynamic in (2.1) if the market is arbitrage free? This answer is $\boxed{\textbf{No}}$. Reason is simple, in this market, we only have one risk free asset, but we don't have any risk asset ($r(t)$ can be regarded as risk asset just like $S(t)$ in Black-Scholes Model, but we can trade the rate $r(t)$). So absolutely, this market is incomplete, we have no way to hedge the risk caused by the Brownian motion, the risk neutral measure is not unique, we can't derive a unique prices for all bonds with different maturities. But it doesn't mean that bond prices can take any form whatsoever. On the contrary we have the following basic intuition:

- Prices of bonds with different maturities will have to satisfy certain internal consistency relations in order to avoid arbitrage opportunities.
- If we take the price of one particular benchmark bond as given, then the prices of all other bonds will be determined in terms of the price of the benchmark bond.

How it works? Because right now, this market consists one benchmark bond plus one risk free asset, then meta-theorem will make sure this market is complete and all assets have unique prices.

2.2 § The Term Structure Equation

To make the ideas presented in the previous section more concrete we now begin with some additional assumptions.

Assumption 2.2.1. *We assume that there is a bonds market for all maturities T and that the market is arbitrage free. We assume furthermore that for every T, the price of a T-bond has the form*

$$p(t,T) = F(t,r(t);T), \tag{2.3}$$

where F is a smooth function of three real variables. And for simplicity, we for a T-bond, we can write its price as $F^T(r,t)$ instead of $p(t,T) = F(t,r(t);T)$.

And to avoid arbitrage, it must hold

$$F(T,r;T) = 1,\ \forall\ T \tag{2.4}$$

Now we applied Ito formula for the T-bond F^T

$$dF^T = F^T\alpha_T dt + F^T\sigma_T dW_t, \tag{2.5}$$

where,

$$\alpha_T = \frac{F_t^T + \mu F_r^T + \frac{1}{2}\sigma^2 F_{rr}^T}{F^T} \tag{2.6}$$

$$\sigma_T = \frac{\sigma F_r^T}{F^T} \tag{2.7}$$

How to judge whether the market is complete? We can apply the **Second Fundament Thoerem** to check whether the risk neutral measure is unique in the market. In other words, the risk neutral measure at time t exists, we first derive the discount value for a T-bond,

$$d\frac{F^T(t,r(t))}{B_t} = \frac{F^T(t,r(t))}{B_t}[(\alpha_T - r(t))dt + \sigma_T dW_t] \tag{2.8}$$

we second apply **Girsanov Theorem** and have,

$$d\frac{F^T(t,r(t))}{B_t} = \frac{F^T(t,r(t))}{B_t}\sigma_T[\frac{\alpha_T(t) - r(t)}{\sigma_T}dt + dW_t] = \frac{F^T(t,r(t))}{B_t}\sigma_T dW_t^*. \tag{2.9}$$

To make the risk neutral measure at time t exist, for any $T \neq S$,

$$\frac{\alpha_T(t) - r(t)}{\sigma_T} = \frac{\alpha_S(t) - r(t)}{\sigma_S}. \tag{2.10}$$

Because $r(t)$ is $\mathcal{F}_t$ measurable, we can write it as $\lambda(t, r(t))$:

Proposition 2.2.1. *Assume that the bond market is free of arbitrage. Then there exists a process λ such that the relation*

$$\frac{\alpha_T(t) - r(t)}{\sigma_T} = \lambda(t, r(t)) \tag{2.11}$$

holds for all t and for every choice of maturity time T.

By equation (2.5), α_T is the local rate of return on the T-bond, where $r(t)$ is the return of risk free asset. And the σ_T is the local volatility of the T-bond. This equation is very similar with the Sharpe Ratio in investment theory, so Proposition 2.2.1 follow this intuition:

- In a no arbitrage market all bonds will have the same Sharpe ratio regardless of maturity time.
- Why this intuition is True? Because of the modern portfolio theory, investors prefer to choose the asset with the highest Sharpe ratio. (This asset is called market portfolio or tangency portfolio in investment theory.) By the Demand-Supply equilibrium in Economics, the market will push the Sharpe ratios of different bonds to the same value $\lambda(t, r(t))$.

Plug (2.6), (2.7) into equation (2.11), we will have the term structure equation.

Proposition 2.2.2. ***(Term Structure Equation)*** *In an arbitrage free market. F^T will satisfy the term structure equation*

$$\begin{cases} F_t^T + (\mu - \lambda\sigma)F_r^T + \frac{1}{2}\sigma^2 F_{rr}^T - rF^T = 0 \\ F^T(T, r) = 1 \end{cases} \tag{2.12}$$

Given the function of $\lambda(t, r)$, we can solve the PDE by risk neutral valuation method.

Proposition 2.2.3. ***(Risk Neutral Valuation)*** *Bond prices are given by the formula $p(t, T) = F(t, r(t); T)$ where*

$$F(t, r; T) = E_{t,r}^Q[e^{-\int_t^T r(s)ds}] \tag{2.13}$$

Here the martingale measure Q denotes the measure where $r(t)$ can be written as

$$dr(t) = (\mu - \lambda\sigma)dt + \sigma dW_t^* \tag{2.14}$$

where W_t^* is the Brownian motion under Q measure.

We see that the value of a T-bond at time t is the expected value of the final payoff of one dollar, discounted to present value under the risk neutral measure Q. And the deflator used is the natural one, namely $e^{-\int_t^T r(s)ds}$. And we can see that choosing different Sharpe ratio will have different risk neutral measure Q.

Recall the Black-Scholes model, which is a complete market model, the martingale measure is uniquely determined. Now the model is not complete, so bond prices will not be uniquely determined by the given (P-)dynamics of the short rate r. Proposition (2.2.1) said the Sharpe ratios of all bonds should be the same in case of arbitrage opportunity, but in this incomplete market, investors can't make an agreement on the unique Sharpe ratio. We can also turn the argument around and say that when the market has determined the dynamics of **one** bond (benchmark) price process, say with maturity T, then the market has indirectly specified λ by equation (2.11), then the model becomes complete. When λ is thus determined, all other bond prices will be determined by the term structure equation.

3 ¶ Martingale Models for the Short Rate

3.1 § Q-dynamics

Recall the short rate P-dynamics is given by

$$dr(t) = \mu(t, r(t))dt + \sigma(t, r(t))dW_t. \tag{3.1}$$

And the term structure equation is

$$\begin{cases} F_t^T + (\mu - \lambda\sigma)F_r^T + \dfrac{1}{2}\sigma^2 F_{rr}^T - rF^T = 0 \\ F^T(T, r) = 1 \end{cases} \tag{3.2}$$

According to what we have talked in last chapter, the short rate model can be also rewritten in Q-dynamic

$$dr(t) = (\mu - \lambda\sigma)dt + \sigma dW_t^*.$$

So instead of specifying μ and λ under the objective probability measure P we will henceforth specify the dynamics of the short rate r directly under the martingale measure Q. This procedure is known as **martingale modeling**, and the typical assumption will thus be that r under Q has dynamics given by

$$dr(t) = \mu(t, r(t))dt + \sigma(t, r(t))dW_t^*. \tag{3.3}$$

From now on the letter μ will thus always denote the drift term of the short rate of interest under the martingale measure Q. (Where $\mu(t, r(t)) = \mu - \lambda\sigma$, and λ can be any legal choice)

3.2 § Short Rate Model

We present a list of the most popular models. If a parameter is time dependent this is written out explicitly. Otherwise all parameters are constant.

1. Vasicek Model

$$dr = (b - ar)dt + \sigma dW_t^*, (a > 0), \tag{3.4}$$

2. Cox-Ingersoll-Ross

$$dr = a(b - r)dt + \sigma\sqrt{r}dW_t^*, \tag{3.5}$$

3. Dothan

$$dr = ardt + \sigma r dW_t^*, \tag{3.6}$$

4. Black-Derman-Toy

$$dr = \theta(t) r dt + \sigma(t) r dW_t^*, \tag{3.7}$$

5. Ho-Lee

$$dr = \theta(t) dt + \sigma dW_t^*, \tag{3.8}$$

6. Hull-White (extended Vasicek)

$$dr = (\theta(t) - a(t) r) dt + \sigma(t) dW_t^*, (a(t) > 0), \tag{3.9}$$

7. Hull-White (extended CIR)

$$dr = (\theta(t) - a(t) r) dt + \sigma(t)\sqrt{r} dW_t^*, (a(t) > 0). \tag{3.10}$$

3.3 § Inversion of the Yield Curve

Now the question is how we estimate the model parameters in the martingale models above? An intuitive procedure is using historical data to estimate the SDE parameters. **But it doesn't work!**

We model the short rate in Q measure, but the historical data we observed is in objective P measure. **But this is not so hopeless!**

- On the one hand, we can use P-data to estimate diffusion parameters in Q-dynamics, because that the Girsanov transformation will only affect the drift term but not the diffusion term.
- So we have to use completely different methods to estimate drift term in Q-dynamics, which is called **inverting the yield curve**
  - Choose a model involving one or several parameters. Let us denote the entire parameter vector by α. Thus we write the r-dynamics under Q as

$$dr(t) = \mu(t, r(t); \alpha) dt + \sigma(t, r(t); \alpha) dW_t^* \tag{3.11}$$

  - For every conceivable time of maturity T, solve the term structure equation

$$\begin{cases} F_t^T + (\mu - \lambda\sigma) F_r^T + \dfrac{1}{2}\sigma^2 F_{rr}^T - rF^T = 0 \\ \hfill F^T(T, r) = 1 \end{cases} \tag{3.12}$$

  - In this way we have computed the theoretical term structure as

$$p(t, T; \alpha) = F^T(t, r; \alpha) \tag{3.13}$$

- Collect the data from the market, the empirical term structure $\{p^*(0,T); T \geq 0\}$. Now choose the parameter vector α^* which makes the theoretical curve $\{p(0,T;\alpha^*), T \geq 0\}$ fits the empirical curve $\{p^*(0,T), T \geq 0\}$ as well as possible
- Now insert α^* into μ and σ, denote our estimation as mu^* and σ^* respectively.

Now we have solve the parameter estimation problems in models above. **But why we want to set the model in that form?** Because under these forms, some of them have recursive property (which is empirical in real world), what's more important is that the PDE will be solved easily, and this leads us to the subject of **affine term structures**.

3.4 § Affine Term Structure

Definition 3.4.1. *If the term structure $\{p(t,T), T \geq t \geq 0\}$ has the form*

$$p(t,T) = F^T(t,r), \tag{3.14}$$

where F has the form

$$F^T(t,r) = e^{A(t,T)-B(t,T)r}, \tag{3.15}$$

*and where A and B are deterministic functions, then the model is said to possess an **affine term structure**.*

A natural question is that how should we choose μ and σ in the Q-dynamics so that can we get a affine term structure?

Assume we have the Q-dynamics

$$dr(t) = \mu(t, r(t))dt + \sigma(t, r(t))dW_t^* \tag{3.16}$$

and we hope the bond prices have the form in (3.15). Apply Ito formula to equation (3.16) and plug it into the term structure equation, we will have

$$A_t(t,T) - (1 + B_t(t,T))r - \mu(t,r)B(t,T) + \frac{1}{2}\sigma^2(t,r)B^2(t,T) = 0. \tag{3.17}$$

And the boundary value $F^T(T,r) \equiv 1$ implies

$$\begin{cases} A(t,T) = 0, \\ B(t,T) = 0 \end{cases} \tag{3.18}$$

Generally speaking, for arbitrary μ and σ, the ODE in (3.17) may have no solution. But luckily, if μ and σ have the form

$$\begin{cases} \mu(t,r) = \alpha(t)r + \beta(t), \\ \sigma(t,r) = \sqrt{\gamma(t)r + \delta(t)}. \end{cases} \tag{3.19}$$

Then, plug (3.19) into (3.17), it transforms into

$$[A_t(t,T)-\beta(t)B(t,T)+\frac{1}{2}\delta(t)B^2(t,T)]-r[1+B_t(t,T)+\alpha(t)B(t,T)-\frac{1}{2}\gamma(t)B^2(t,T)] \tag{3.20}$$

The equation holds for all t, T, r. So for a fxied choice of T and t, we will have system

$$\begin{cases} A_t(t,T) - \beta(t)B(t,T) + \frac{1}{2}\delta(t)B^2(t,T) = 0, \\ 1 + B_t(t,T) + \alpha(t)B(t,T) - \frac{1}{2}\gamma(t)B^2(t,T) = 0. \end{cases} \tag{3.21}$$

On the textbook page 331, it talked some reasons why those models fancy. For instance, the short rates in Vasicek, Ho-Lee and Hull-White are linear SDEs, so they are easy to solve and will be normally distributed. But they also have defect, the interest rate may be negative due to the normality. So in Dothan, CIR and Hull-White extension model, we make a trade-off between keeping the short rate always be positive and computation complex.

Another problem is the parameter estimation. In real world, we can observe infinite bond prices $\{p^*(0,T), for\ all\ T>0\}$, but in the models with only finite unknown parameters (like Vasicek model), we may not get a nice parameter estimation.So Hull-White introduce the model with infinite parameters (like $a(t)$, which is time dependent), then we may get a perfect fit, but it may also cause the over-fitted problem.

3.5 § Some Standard Models

I will skip this section, check page 332 for reference.

4 ¶ Forward Rates Model

4.1 § Heath-Jarrow-Morton Framework

In last chapter, we use the short rate r as the only variable of the T-bond prices. The main advantages are as follows:

- Specifying r as the solution of an SDE allows us to use Markov process theory, so we may work within a PDE framework.
- In particular it is often possible to obtain analytical formulas for bond prices and derivatives.

The main drawbacks of short rate models are as follows:

- From an economic point of view it is unreasonable to assume that the entire money market is governed by only one explanatory variable.
- As the short rate model becomes more realistic, the inversion of the yield curve described above becomes increasingly more difficult.

To solve this issue, the method proposed by Health-Jarrow-Morton (HJM), they choose the entire forward rate curve as their (infinite dimensional) state variable. We now turn to the specification of the HJM framework.

Assumption 4.1.1. *We assume that, for every fixed $T > 0$, the forward rate f(*,T) has a stochastic differential which under the objective measure P is given by*

$$df(t,T) = \alpha(t,T)dt + \sigma(t,T)dW_t \tag{4.1}$$

$$f(0,T) = f^*(0,T) \tag{4.2}$$

where W is a P-Browinian motion whereas $\alpha(t,T)$ and $\sigma(t,T)$ are adapted processes. And note that we use the observed forward rate curve $\{f^*(0,T); T \geq 0\}$ as th initial condition.

$\boxed{\textit{Remark}}$: It is important to observe that the HJM is not a specific model (like the Vasicek model). It is a framework to be used for analyzing interest rate models. I wrote an article about this framework, here is the link .